

# Review of Decoherence Free Subspaces, Noiseless Subsystems, and Dynamical Decoupling

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Quantum information requires protection from the adverse affects of decoherence and noise. This review provides an introduction to the theory of decoherence-free subspaces, noiseless subsystems, and dynamical decoupling. It addresses quantum information preservation as well as protected computation.

## I. INTRODUCTION

The protection of quantum information is a central task in quantum information processing [1]. Decoherence and noise are obstacles which must be overcome and managed before large scale quantum computers can be built. This review provides an introduction to the theory of decoherence-free subspaces, noiseless subsystems, and dynamical decoupling, among the key tools in the arsenal of decoherence mitigation strategies. It is based on lectures given by the author at the University of Southern California as part of a graduate course on quantum error correction, and as such is not meant to be a comprehensive review, nor to supply an exhaustive list of references. Rather, the goal is to get the reader quickly up to speed on a subset of key topics in the field of quantum noise avoidance and suppression. For previous reviews overlapping with some of the theoretical topics covered here see, e.g., Refs. [2, 3].

The review is structured as follows. Section II introduces decoherence-free subspaces (DFSs). Section III defines and analyzes the collective dephasing model, and explains how to combine the corresponding DFS encoding with universal quantum computation. Section IV considers the same problem in the context of the more general collective decoherence model, where general noise afflicts all qubits simultaneously. Section V introduces and analyzes noiseless subsystems (NSs), a key generalization of DFSs which underlies all known methods of quantum information protection. The NS structure is illustrated with the three-qubit code against collective decoherence, including computation over this code. We then proceed to dynamical decoupling (DD). Section VI introduces the topic by analyzing the protection of a single qubit against pure dephasing and against general decoherence, using both ideal (zero-width) and real (finite-width) pulses. Section VII briefly discusses DD as a symmetrization procedure. Section VIII discusses combining DD with DFS in the case of two qubits. Section IX addresses concatenated dynamical decoupling (CDD), a method to achieve high-order decoupling. In the final technical Section X, we come full circle and connect dynamical decoupling to the representation theory ideas underlying noiseless subsystems theory, thus presenting a unified view of the the approaches. Concluding remarks and additional literature entries are presented in Section XI.

## II. DECOHERENCE-FREE SUBSPACES

Let us begin by assuming we have two systems:  $A$  and  $B$ , defined by the Hilbert spaces  $\mathcal{H}_A$  and  $\mathcal{H}_B$ , respectively. In general, the dynamics of these two systems are governed by

$$H = H_A + H_B + H_{AB}, \quad (1)$$

where  $H_A$  and  $H_B$  correspond to the pure dynamics of system  $A$  and  $B$ , respectively, and  $H_{AB}$  is the interaction between the two systems. For simplicity assume  $H_A \equiv 0$  and  $H_B \equiv 0$ , and let us focus on the dynamics of  $A$  resulting from the interaction Hamiltonian. Recall that using the Kraus representation [1], we can effectively study the reduced dynamics of  $A$  for an initial state  $\rho_A(0)$ , where

$$\rho_A(0) \mapsto \rho_A(t) = \sum_{\alpha} K_{\alpha}(t) \rho_A(0) K_{\alpha}^{\dagger}(t) \quad (2)$$

after the partial trace over system  $B$  is completed. The Kraus operators  $K_{\alpha}(t)$  satisfy the relation  $\sum_{\alpha} K_{\alpha}^{\dagger}(t) K_{\alpha}(t) = I_A \forall t$ , where  $I_A$  is the Identity operator on system  $A$ . In general, we see that the Kraus operators result in non-unitary evolution in the system  $A$  Hilbert space. Therefore, let us define decoherence as follows:

**Definition 1.** *A open system undergoes decoherence if its evolution is not unitary. Conversely, an open system which undergoes purely unitary evolution is said to be decoherence-free.*

It is from this definition of decoherence that a DFS will be defined for system  $A$ .

Since the DFS defines a portion of the system Hilbert space where decoherence is not present, let us partition the Hilbert space into two subspaces  $\mathcal{H}_A = \mathcal{H}_G \oplus \mathcal{H}_N$ . The section of the Hilbert space not effected is denoted by  $\mathcal{H}_G$ , the “good” portion,

and  $\mathcal{H}_N$  denotes the decoherence-effected Hilbert space. From the definition of decoherence we find that if  $\mathcal{H}_A$  is to remain uneffected it must evolve unitarily. Therefore, assume that it is possible to partition the Kraus operators as

$$K_\alpha(t) = g_\alpha U \oplus B_\alpha \quad (3)$$

such that  $U$  defines a unitary operator acting on  $\mathcal{H}_G$ ,  $g_\alpha \in \mathbb{C}$ , and  $B_\alpha$  is an arbitrary (possibly non-unitary) operator acting solely on  $\mathcal{H}_N$ . Note that by the completeness relation of the Kraus operators,

$$\sum_\alpha |g_\alpha|^2 = 1, \quad \sum_\alpha B_\alpha^\dagger B_\alpha = I_N. \quad (4)$$

Assume that the initial state is partitioned as well, as

$$\rho_A(0) = \rho_G(0) \oplus \rho_N(0). \quad (5)$$

In terms of the newly partitioned Kraus operators Eq. (2) then becomes

$$\rho_A \mapsto \rho'_A = \sum_\alpha (g_\alpha U \oplus B_\alpha) \rho_A(0) (g_\alpha^* U^\dagger \oplus B_\alpha^\dagger) = \begin{pmatrix} U \rho_G(0) U^\dagger & 0 \\ 0 & \sum_\alpha B_\alpha \rho_N(0) B_\alpha^\dagger \end{pmatrix}. \quad (6)$$

The remarkable thing to notice about this last result is that  $\rho_G$  evolves purely unitarily, i.e., it satisfies the definition of decoherence-freeness. Thus, we conclude that if the Kraus operators have a block diagonal form as in Eq. (3) then a DFS  $\mathcal{H}_G$  exists.

### A. Deutsch's Algorithm

As a first example of error avoidance utilizing the DFS construction, we can consider the first known algorithm that offers a quantum speed-up. Deutsch's Algorithm [1] presents a simple decision problem in which the goal is to decide whether a function is constant or balanced. Let  $f(x) : \{0, 1\}^n \mapsto \{0, 1\}$  (in decimal notation  $x \in \{0, 1, \dots, 2^n - 1\}$ ), denote the function, where if

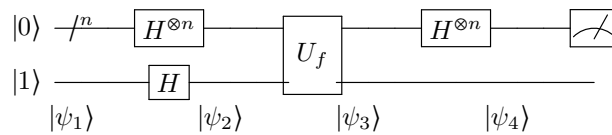
$$f(x) = \begin{cases} 0, & \forall x \text{ or} \\ 1, & \forall x \end{cases} \quad (7)$$

then the function is called constant and if

$$f(x) = \begin{cases} 0, & \text{half the inputs} \\ 1, & \text{half the inputs} \end{cases} \quad (8)$$

then the function is called balanced. Classically, we find that in the worst case, making a decision on whether  $f(x)$  is constant or balanced requires a minimum of  $2^n/2 + 1$  total queries to  $f$ . Deutsch and Jozsa found that the exponential cost in  $f$ -queries is drastically reduced by considering a quantum version of the algorithm.

The equivalent version of the decision problem can be recast in terms of the quantum circuit:



where each classical bit  $n$  corresponds to a qubit. The unitary operator  $U$  performs the query on  $f(x)$  by

$$U_f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle \text{ (addition mod 2)} \quad (9)$$

and  $H$  represents the Hadamard gate:  $H|0\rangle = |+\rangle$  and  $H|1\rangle = |-\rangle$ , where  $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ .

In order to illustrate Deutsch's algorithm for the quantum circuit above, consider the single qubit version ( $n = 1$ ). In this case there are four functions, two of which are constant and two of which are balanced:  $\{f_0(x) = 0, f_1(x) = 1\}$  (constant),  $\{f_2(x) = x, f_3(x) = \bar{x}\}$  (balanced), where the bar denotes bit negation. Clearly two classical queries to  $f$  are required to tell whether  $f$  is constant or balanced.

Initially the total system state is given by  $|\psi_1\rangle = |0\rangle|1\rangle$ . Applying the Hadamard gate, the system state becomes  $|\psi_2\rangle = |+\rangle|-\rangle$ . Applying the unitary operator  $U_f$ , the resulting state is

$$|\psi_3\rangle = U_f |+\rangle|-\rangle = U_f \left[ \frac{1}{2} (|00\rangle - |01\rangle + |10\rangle - |11\rangle) \right] \quad (10a)$$

$$= \frac{1}{2} (|0, 0 \oplus f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, 0 \oplus f(1)\rangle - |1, 1 \oplus f(1)\rangle) \quad (10b)$$

$$= \frac{1}{2} (|0, f(0)\rangle - |0, \bar{f}(0)\rangle + |1, f(1)\rangle - |1, \bar{f}(1)\rangle). \quad (10c)$$

Applying each constant and balanced function to  $|\psi_3\rangle$ , we find

$$|\psi_3\rangle = \begin{cases} f_0 : |00\rangle - |01\rangle + |10\rangle - |11\rangle = +|+-\rangle \\ f_1 : |01\rangle - |00\rangle + |11\rangle - |10\rangle = -|+-\rangle \\ f_2 : |00\rangle - |01\rangle + |11\rangle - |10\rangle = +|--\rangle \\ f_3 : |01\rangle - |00\rangle + |10\rangle - |11\rangle = -|--\rangle \end{cases} \quad (11)$$

for  $f_j(x) \in \{0, 1, x, \bar{x}\}$  as defined by Eqs. (7) and (8). The remaining Hadamard gate yields the final state

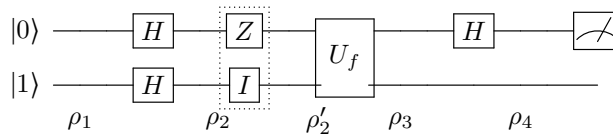
$$|\psi_4\rangle = \begin{cases} f_0 : +|0-\rangle \\ f_1 : -|0-\rangle \\ f_2 : +|1-\rangle \\ f_3 : -|1-\rangle \end{cases} \quad (12)$$

and the characteristic of the function is determined by measuring the 1st qubit: a result of 0 indicates a constant function, a result of 1 a balanced function. Thus, remarkably, we find that the quantum version only requires a *single* query to the function, while the classical case requires two queries (this scenario is the original Deutsch algorithm).

The circuit depicted above can be subjected to a similar analysis in the  $n$ -qubit case (the Deutsch-Jozsa algorithm) and the conclusion is that the quantum version of the algorithm still requires only a single  $f$ -query, thus resulting in an exponential speed-up relative to its classical counterpart in the worst case.

## B. Deutsch's Algorithm With Decoherence

To gain an understanding of how a DFS works we can look at the Deutsch Problem with added decoherence. We can consider the circuit diagram for the single qubit Deutsch algorithm, but introduce a dephasing element as follows, where the dotted box denotes dephasing on the top qubit only:



The Kraus Operators governing the dephasing of  $\rho_2$  are:

$$K_0 = \sqrt{1-p} I_1 \otimes I_2 \quad (13a)$$

$$K_1 = \sqrt{p} Z_1 \otimes I_2 \quad (13b)$$

With probability  $(1-p)$  nothing happens. However with probability  $p$  the first qubit experiences dephasing. If we follow the density matrix states through the algorithm we can see the effect this dephasing has on our result. As before we have  $\rho_1 = |01\rangle\langle 01|$  and,  $\rho_2 = |+-\rangle\langle +-|$ . By applying the Kraus operators we find the state after dephasing to be

$$\rho'_2 = K_0 \rho_2 K_0^\dagger + K_1 \rho_2 K_1^\dagger \quad (14a)$$

$$= (1-p) \rho_2 + p (Z \otimes I) |+-\rangle\langle +-| (Z \otimes I)^\dagger. \quad (14b)$$

It is easy to check that  $Z|+\rangle = |-\rangle$ , thus we find

$$\rho'_2 = (1-p) |+-\rangle\langle +-| + p |--\rangle\langle --|. \quad (15)$$

Using Eq. (9) we can compute  $\rho_3$

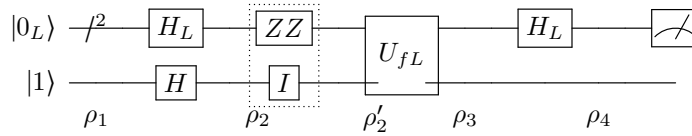
$$\rho_3 = \begin{cases} (1-p)|+-\rangle\langle+-| + p|--\rangle\langle--| & f_0, f_1 \text{ (constant)} \\ (1-p)|--\rangle\langle--| + p|+-\rangle\langle+-| & f_2, f_3 \text{ (balanced)} \end{cases} \quad (16)$$

After applying the final Hadamard we see that the state we will measure is

$$\rho_4 = \begin{cases} (1-p)|0-\rangle\langle0-| + p|1-\rangle\langle1-| & f_0, f_1 \text{ (constant)} \\ (1-p)|1-\rangle\langle1-| + p|0-\rangle\langle0-| & f_2, f_3 \text{ (balanced)} \end{cases} \quad (17)$$

If we now measure the first qubit to determine whether the function is constant or balanced, with probability  $p$  we will misidentify the outcome. For example, if we obtain the outcome 1, with probability  $p$  this could have come from the constant case. But, according to Eq. (12) the outcome 1 belongs to the balanced case.

It is possible to overcome this problem by use of a DFS. Let us again modify the original circuit design. We can add a third qubit and then define logical bits and gates.



Now the  $Z$  dephasing acts simultaneously on both top qubits, that comprise the logical qubit  $|0_L\rangle$ . In this case the Kraus operators are

$$K_0 = \sqrt{1-p}III, \quad K_1 = \sqrt{p}ZZI, \quad (18)$$

where  $ZZI = \sigma^z \otimes \sigma^z \otimes I$ . Recall the requirements for a DFS. The Kraus operators, as in Eq. (3), must be of the form

$$K_\alpha = \left( \begin{array}{c|c} g_\alpha U & 0 \\ \hline 0 & B_\alpha \end{array} \right) \quad (19)$$

and the state must be initialized in a *good* subspace, i.e.,  $\rho_S = \rho_G \oplus \rho_N$ , where the direct sum reflects the same block structure as in Eq. (19). If these conditions are met then  $\rho'_S = \sum K_\alpha \rho_S K_\alpha^\dagger = U \rho_G U^\dagger \oplus \rho'_N$ . In other words the evolution of  $\rho_G$  is entirely unitary. We start by checking the matrix form of the Kraus operators.  $K_0$  is simply the identity and trivially satisfies this condition. We can check the  $ZZ$  portion of  $K_1$  since that is what will act on our logical qubit.

$$ZZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}. \quad (20)$$

This obviously doesn't fit the required matrix format, in that there is no block of 1's like in  $K_0$ . However with a simple reordering of the basis states we obtain the following matrix

$$ZZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} 00 \\ 11 \\ 01 \\ 10 \end{matrix}. \quad (21)$$

Now we have a  $2 \times 2$  matrix of 1's, so both  $ZZ$  and the identity matrix act as the same unitary on the subspace spanned by  $|00\rangle$  and  $|11\rangle$ . The full matrix,  $K_1$ , then takes the form

$$K_1 = \sqrt{p} \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & I_2 \end{pmatrix} \begin{matrix} 00i \\ 11i \\ 01i \\ 10i \end{matrix} \quad (22)$$

where  $I_2$  denotes the  $2 \times 2$  identity matrix,  $i = \{0, 1\}$ , and  $K_0$  is the  $8 \times 8$  identity matrix. Thus we see that both  $K_0$  and  $K_1$  have the same upper block format, namely  $U = I_{4 \times 4}$ , where  $g_0 = \sqrt{1-p}$  and  $g_1 = \sqrt{p}$ . Now we can define our logical bits  $|0_L\rangle = |00\rangle$  and  $|1_L\rangle = |11\rangle$ . With these states we can construct our logical Hadamard.

$$(H_L)_{4 \times 4} = \left( \begin{array}{c|c} H & 0 \\ \hline 0 & V \end{array} \right). \quad (23)$$

Here the logical Hadamard acts as a regular Hadamard on our logical qubits.

$$H_L |0_L\rangle = |+_L\rangle \quad (24a)$$

$$H_L |1_L\rangle = |-_L\rangle \quad (24b)$$

where  $|\pm_L\rangle = \frac{1}{\sqrt{2}} (|0_L\rangle \pm |1_L\rangle)$ . The other unitary action of  $H_L$ , namely  $V$ , we don't care about. Similarly we can construct a logical  $U_f$ .

$$(U_{fL})_{8 \times 8} = \left( \begin{array}{c|c} U_f & 0 \\ \hline 0 & W \end{array} \right). \quad (25)$$

Again,  $U_{fL}$  acts as  $U_f$  on our logical bits, and  $W$  we don't care about. Neither  $V$  nor  $W$  affect our logical qubits in any way. Now that we have set our system up we can apply the Deutsch algorithm again to see if the DFS corrects the possibility of misidentifying the result. Our system begins in the state  $\rho_1 = |0_L\rangle|1\rangle\langle 0_L|\langle 1|$  and after applying the logical Hadamard we get  $\rho_2 = |+_L\rangle|-_L\rangle\langle +_L|\langle -_L|$ . Now we can apply the Kraus operators to see the effect of the decoherence.  $K_0$  has no effect other than to multiply the state by  $\sqrt{1-p}$  because it is proportional to the identity matrix. It is enough to examine the effect of  $K_1$  on the state  $|+_L\rangle$ .

$$K_1 |+_L\rangle = \frac{1}{\sqrt{2}} (K_1 |0_L\rangle + K_1 |1_L\rangle) \quad (26a)$$

$$= \frac{1}{\sqrt{2}} \sqrt{p} (I |0_L\rangle + I |1_L\rangle) \quad (26b)$$

$$= \sqrt{p} |+_L\rangle \quad (26c)$$

Therefore  $\rho'_2 = \rho_2$ . The decoherence has no effect on our system and the rest of the algorithm will proceed without any possibility of error in the end.

### C. A Classical Example

We can look at a classical case to better understand this effect. Assume we have three parties: Alice, Bob, and Eve. Alice wants to send a message to Bob, and Eve wants to mess that message up. Let's also assume that the only way in which Eve can act to mess up the message is by, with some probability, flipping all of the bits of the message. If Alice were to send only one bit to Bob there would be no way of knowing if that bit had been flipped. But let's say Alice is smarter than Eve and decides to send two bits. She also communicates with Bob before hand and tells him that if he receives a 00 or 11 he should treat it as a logical 0, and if he receives a 01 or 10 to treat it as a logical 1. If this scheme is used, Eve's ability to flip both bits has no effect on their ability to communicate.

$$0_L = \begin{cases} 00 \\ 11 \end{cases} \rightarrow \begin{cases} 11 \\ 00 \end{cases} = 0_L \quad (27a)$$

$$1_L = \begin{cases} 01 \\ 10 \end{cases} \rightarrow \begin{cases} 10 \\ 01 \end{cases} = 1_L \quad (27b)$$

This is an example of parity conservation. The logical 0 is even parity and the logical 1 is odd parity. Encoding logical bits in parity in this way effectively hides the information from Eve's bit flip error.

### D. Hamiltonian DFS

Discussing DFS in terms of Kraus operators works well, but we'd like to develop a bottom-up understanding of the DFS concept, using Hamiltonian evolution. Assume we are given a system in which our computation is occurring, and a bath that is

connected to the system. The Hamiltonian governing the whole system can be written as usual as

$$H = H_S \otimes I_B + I_S \otimes H_B + H_{SB}, \quad (28)$$

where  $H_S$  acts only on the system we are interested in,  $H_B$  acts only on the bath, and  $H_{SB}$  governs the interaction between the two. Assume also, without loss of generality, that the interaction Hamiltonian can be written as

$$H_{SB} = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}, \quad (29)$$

where each  $S_{\alpha}$  is a pure-system operator and each  $B_{\alpha}$  is a pure-bath operator. The Hilbert space can be written  $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ , and  $\mathcal{H}_S = \mathcal{H}_G \otimes \mathcal{H}_N$  where

$$\mathcal{H}_G = \text{Span}\{|\gamma_i\rangle\} \quad (30a)$$

$$\mathcal{H}_N = \text{Span}\{|\nu_k\rangle\} \quad (30b)$$

$$\mathcal{H}_B = \text{Span}\{|\beta_j\rangle\} \quad (30c)$$

To formulate a theorem we also need the following assumptions:

1. The system state is initialized in the *good* subspace:

$$\rho_S = \rho_G = \sum_{i,j} r_{ij} |\gamma_i\rangle \langle \gamma_j| \quad (31)$$

2. The basis states of the good subspace are eigenvectors of the interaction Hamiltonian:

$$S_{\alpha} |\gamma_i\rangle = c_{\alpha} |\gamma_i\rangle, c_{\alpha} \in \mathbb{C} \quad (32)$$

3. The basis states of the good subspace, when acted on by the system Hamiltonian, remain in the good subspace:

$$H_S |\gamma_i\rangle \in \mathcal{H}_G \quad (33)$$

With these assumptions in hand we can posit the following theorem.

**Theorem 1.** *Assuming 1-3, the evolution of the system,  $\rho_S$ , can be written as*

$$\text{Tr}_B[U(\rho_S(0) \otimes \rho_B(0))U^{\dagger}] = U_S \rho_G(0) U_S^{\dagger} \quad (34a)$$

$$U_S |\gamma_i\rangle \in \mathcal{H}_G \quad (34b)$$

where  $U = U(t) = e^{-iHt}$ .

*Proof.* Using equations (29) and (32) we can write

$$(I_S \otimes H_B + H_{SB}) |\gamma_i\rangle_S \otimes |\beta_j\rangle_B = |\gamma_i\rangle \otimes (H_B |\beta_j\rangle) + \sum_{\alpha} c_{\alpha} |\gamma_i\rangle \otimes B_{\alpha} |\beta_j\rangle \quad (35a)$$

$$= |\gamma_i\rangle \otimes \left( \sum_{\alpha} c_{\alpha} B_{\alpha} + H_B \right) |\beta_j\rangle \quad (35b)$$

$$= |\gamma_i\rangle \otimes H_C |\beta_j\rangle, \quad (35c)$$

where  $H_C$  acts only on the bath. Applying this to equation (28) we find that the complete Hamiltonian can be decomposed into a portion that acts only on the system and a portion that acts only on the bath.

$$H |\gamma_i\rangle \otimes |\beta_j\rangle = (H_S \otimes I_B + I_S \otimes H_C) |\gamma_i\rangle \otimes |\beta_j\rangle. \quad (36)$$

If we plug this form of the Hamiltonian into the unitary evolution matrix we get

$$U(t) = e^{-i(H_S \otimes I_B + I_S \otimes H_C)t} = U_S(t) \otimes U_C(t), \quad (37)$$

where  $U_x(t) = \exp(-itH_x)$ ,  $x = S, C$ .

To find  $\rho(t)$  we apply this unitary to  $\rho(0)$  with  $\rho_S(0) = \rho_G(0)$ .

$$\rho(t) = U(\rho_S(0) \otimes \rho_B(0))U^{\dagger} \quad (38a)$$

$$= U_S \rho_G(0) U_S^{\dagger} \otimes U_C \rho_B(0) U_C^{\dagger} \quad (38b)$$

We find the state of our system of interest  $\rho_S(t)$  by taking the partial trace.

$$\rho_S(t) = \text{Tr}_B[U_S \rho_G(0) U_S^\dagger \otimes U_C \rho_B(0) U_C^\dagger] \quad (39a)$$

$$= U_S \rho_G(0) U_S^\dagger \quad (39b)$$

□

Thus Theorem 1 guarantees that if its conditions are satisfied, a state initialized in the DFS will evolve unitarily.

### III. COLLECTIVE DEPHASING

#### A. The model

Consider the example of a spin-boson Hamiltonian. In this example, the system of qubits could be the spins of  $N$  electrons trapped in the periodic potential well of a crystalline lattice. The bath is the phonons of the crystal (its vibrational modes). We also assume that the system-bath interaction has some symmetry in the sense that the interaction between the spins and phonons is the same for all spins, e.g., because the phonon wavelength is long compared to the spacing between spins. This assumption is crucial for our purpose of demonstrating the appearance of a DFS. If the potential wells are deep enough then the motional degrees of freedom of the electrons can be ignored. Let  $i$  denote the index for the set of  $N$  electrons in the system (the same as the index for the set of occupied potential wells in the solid), let  $k$  denote the vibrational mode index,  $b_k^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$  is the action of the creation operator for mode  $k$  on a Fock state with occupation number  $n$ ,  $b_k |n\rangle = \sqrt{n} |n-1\rangle$  is the action of the annihilation operator for mode  $k$ . With  $\sigma_i^z$  the Pauli- $z$  spin operator acting on the  $i^{\text{th}}$  spin, the system-bath Hamiltonian is

$$H_{SB} = \sum_{i,k} g_{i,k}^z \sigma_i^z \otimes (b_k + b_k^\dagger). \quad (40)$$

The symmetry assumption implies

$$g_{i,k}^z = g_k^z, \quad (41)$$

i.e., the coupling constants do not depend on the qubit index. The system-bath Hamiltonian can then be written

$$H_{SB} = \sum_i \sigma_i^z \otimes \sum_k g_k^z \cdot (b_k + b_k^\dagger) \quad (42a)$$

$$= S_z \otimes B_z, \quad (42b)$$

where

$$S_z \equiv \sum_i \sigma_i^z, \quad B_z \equiv \sum_k g_k^z (b_k + b_k^\dagger). \quad (43)$$

If these conditions are met, the bath acts identically on all qubits and system-bath Hamiltonian is invariant under permutations of the qubits' order. The operator  $S_z$  is a *collective* spin operator.

#### B. The DFS

Now consider the case of  $N = 2$  in light of the DFS condition Eq. (32):

$$N = 2 \Rightarrow S_z = Z \otimes I + I \otimes Z. \quad (44)$$

Thus

$$\begin{aligned} |00\rangle &\xrightarrow{S_z} 2 \cdot |00\rangle && \Rightarrow c_z = 2 \\ |01\rangle &\xrightarrow{S_z} |01\rangle - |01\rangle = 0 && \Rightarrow c_z = 0 \\ |10\rangle &\xrightarrow{S_z} |10\rangle - |10\rangle = 0 && \Rightarrow c_z = 0 \\ |11\rangle &\xrightarrow{S_z} -2 \cdot |11\rangle && \Rightarrow c_z = -2 \end{aligned} \quad (45)$$

It follows that the DFS's for the two spins are:

$$\begin{aligned}\tilde{\mathcal{H}}_{N=2}(2) &= \{|00\rangle\} \\ \tilde{\mathcal{H}}_{N=2}(0) &= \{|01\rangle, |10\rangle\} \\ \tilde{\mathcal{H}}_{N=2}(-2) &= \{|11\rangle\}\end{aligned}\tag{46}$$

where we used the notation  $\tilde{\mathcal{H}}_N(c_z)$  to denote the “good” subspace  $\mathcal{H}_G$  for  $N$  qubits, with eigenvalue  $c_z$ .

In the  $\tilde{\mathcal{H}}_{N=2}(0)$  DFS, there are two states, so we have an encoded qubit:

$$|\bar{0}\rangle = |01\rangle \quad \text{logical 0} \tag{47a}$$

$$|\bar{1}\rangle = |10\rangle \quad \text{logical 1} \tag{47b}$$

For three spins we have:

$$N = 3 \Rightarrow S_z = Z \otimes I \otimes I + I \otimes Z \otimes I + I \otimes I \otimes Z \tag{48}$$

Similarly, it follows that the DFS's for the three spins are:

$$\begin{aligned}\tilde{\mathcal{H}}_{N=3}(3) &= \{|000\rangle\} \\ \tilde{\mathcal{H}}_{N=3}(1) &= \{|001\rangle, |010\rangle, |100\rangle\} \\ \tilde{\mathcal{H}}_{N=3}(-1) &= \{|011\rangle, |101\rangle, |110\rangle\} \\ \tilde{\mathcal{H}}_{N=3}(-3) &= \{|111\rangle\}\end{aligned}\tag{49}$$

We find that there are two possible encoded qutrits for  $N = 3$ , one in  $\tilde{\mathcal{H}}_{N=3}(1)$  and the other in  $\tilde{\mathcal{H}}_{N=3}(-1)$ .

In general, the DFS  $\tilde{\mathcal{H}}_N(\lambda)$  is the eigenspace of each eigenvalue ( $\lambda = c_z$ ) of  $S_z$ . It is easy to see that the number of spin-ups (0's) and the number of spin-downs (1's) in each eigenstate is constant throughout a given eigenspace. This corresponds to the value of total spin projection along  $z$ . In fact, for arbitrary  $N$ ,

$$c_z = \#0 - \#1. \tag{50}$$

Figure 1, known as the *Bratteli Diagram*, shows the eigenvalues ( $y$ -axis) of  $S_z$  as  $N$  ( $x$ -axis) increases. It represents the constellation of DF subspaces in the parameter-space,  $(N, \lambda)$ . Each intersection point in the figure represents a DFS. Each upward stroke on the diagram indicates the addition of one new spin-up particle,  $|0\rangle$ , to the system, while each downward stroke indicates the addition of a new spin-down particle,  $|1\rangle$ , to the system. From this we see that the dimension of the eigenspace with eigenvalue  $\lambda$  is given by the relation:

$$\dim(\tilde{\mathcal{H}}_N(c_z)) = \binom{N}{\#0}. \tag{51}$$

The highest dimensional DFS for each  $N$  is thus

$$\max_{c_z} \{\dim(\tilde{\mathcal{H}}_N(c_z))\} = \begin{cases} \binom{N}{\frac{N}{2}} & N \text{ even} \\ \binom{N}{\frac{N}{2}+1} & N \text{ odd} \end{cases} \tag{52}$$

Given a  $D$ -dimensional DFS,  $\tilde{\mathcal{H}}$ :

$$\# \text{ of DFS qubits in } \tilde{\mathcal{H}} = \log_2 D. \tag{53}$$

This formula should be thought of as giving the rate of the DFS code, i.e.,

$$r \equiv \frac{\# \text{ of DFS qubits in } \tilde{\mathcal{H}}}{\# \text{ of physical qubits}} = \frac{\log_2 D}{N}. \tag{54}$$

For a DFS of given dimension it may be preferable to think in terms of qudits rather than qubits. For example, for the DFS  $\tilde{\mathcal{H}}_3(-1)$ , the dimension  $D = 3$ , and so this DFS encodes one qutrit.

Any superposition of states in the same DFS/eigenspace will remain in that particular DFS. But a superposition of states in different eigenspaces will not evolve in a decoherence-free manner.



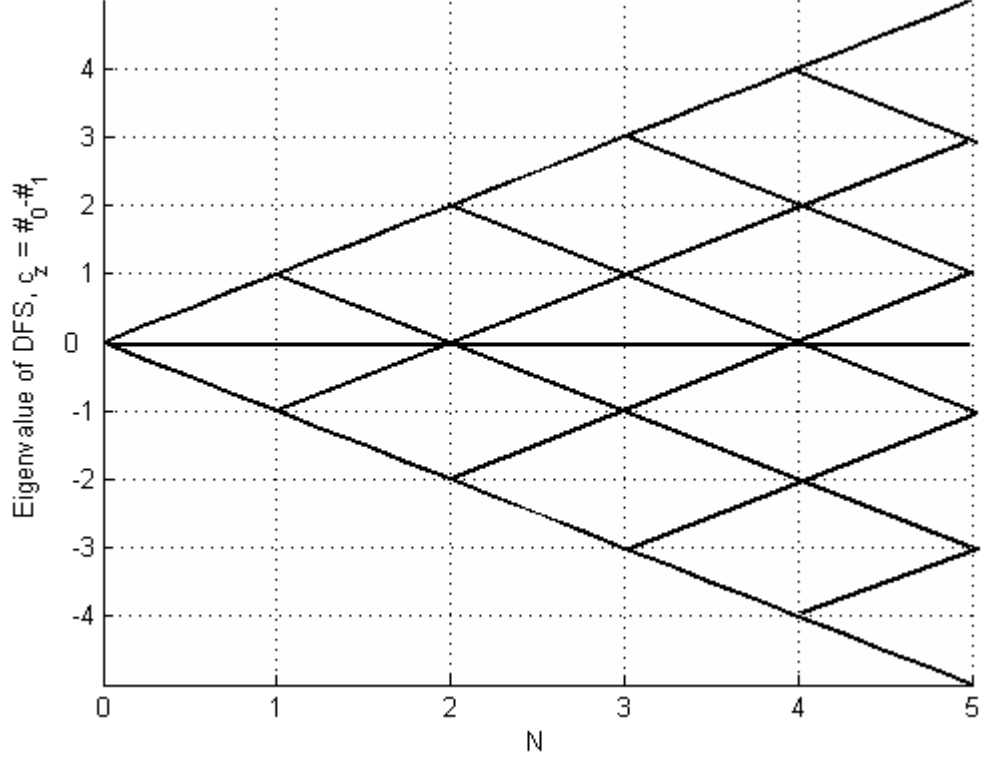


FIG. 1. Bratteli diagram showing DFSs for each  $N$ . Each intersection point corresponds to a DFS. The number of paths leading to a given point is the dimension of the corresponding DFS. The diagram shows, e.g., that there is a 2-dimensional DFS for  $N = 2$  at  $c_z = 0$ , yielding one encoded qubit.

### C. Universal Encoded Quantum Computation

From here on, encoded qubits will be called ‘logical qubits’. To perform arbitrary single qubit operations, we need to be able to apply any two of the Pauli operators on the logical qubits. On our example system,  $\mathcal{H}_2(0)$ , we have:

$$\begin{aligned} |\bar{0}\rangle &\equiv |01\rangle \\ |\bar{1}\rangle &\equiv |10\rangle \\ |\bar{\phi}\rangle &: \text{DFS encoded logical qubit, } a|\bar{0}\rangle + b|\bar{1}\rangle \\ \bar{U} &: \text{Logical operator on the DFS qubits} \end{aligned}$$

Thus the logical Pauli- $z$  operator is

$$\left. \begin{aligned} \bar{Z}|\bar{0}\rangle &= |\bar{0}\rangle \\ \bar{Z}|\bar{1}\rangle &= -|\bar{1}\rangle \end{aligned} \right\} \Rightarrow \bar{Z} = Z \otimes I \Rightarrow \begin{cases} |01\rangle \xrightarrow{Z \otimes I} |01\rangle = |\bar{0}\rangle \\ |10\rangle \xrightarrow{Z \otimes I} -|10\rangle = -|\bar{1}\rangle \end{cases}, \quad (55)$$

and the logical Pauli- $x$  operator is

$$\left. \begin{aligned} \bar{X}|\bar{0}\rangle &= |\bar{1}\rangle \\ \bar{X}|\bar{1}\rangle &= |\bar{0}\rangle \end{aligned} \right\} \Rightarrow \bar{X} = X \otimes X \Rightarrow \begin{cases} |01\rangle \xrightarrow{X \otimes X} |10\rangle = |\bar{1}\rangle \\ |10\rangle \xrightarrow{X \otimes X} |01\rangle = |\bar{0}\rangle \end{cases}. \quad (56)$$

In general, suppose we have  $2N$  physical qubits all experiencing collective dephasing. We can pair them into  $N$  logical qubits, each in  $\mathcal{H}_2(0)$ , and perform  $Z$  or  $X$  logical operations on the  $i^{\text{th}}$  logical qubit using the following operators:

$$\bar{Z}_i \equiv Z_{2i-1} \otimes I_{2i}, \quad (57a)$$

$$\bar{X}_i \equiv X_{2i-1} \otimes X_{2i}. \quad (57b)$$

We can define arbitrary rotations about the logical  $X$  or  $Z$  axis as  $R_{\bar{X}}(\theta) = \exp[i\bar{X}\theta]$  and  $R_{\bar{Z}}(\phi) = \exp[i\bar{Z}\phi]$ . An arbitrary single logical qubit rotation (an arbitrary element of  $SU(2)$ ) can then be obtained using the Euler angle formula, as a product of three rotations:  $R_{\bar{X}}(\theta_2)R_{\bar{Z}}(\phi)R_{\bar{X}}(\theta_1)$ .

To generate arbitrary operators on multiple qubits, we need to add another gate to the generating set: the controlled phase gate. The logical controlled phase gate can be generated from  $\bar{Z}_i \otimes \bar{Z}_j \equiv Z_i \otimes Z_{2j-1}$ . Thus a Hamiltonian of the form

$$\bar{H}_S = \sum_i \omega_{Z_i}(t) \bar{Z}_i + \sum_i \omega_{X_i}(t) \bar{X}_i + \sum_{i \neq j} \Omega_{ij}(t) \bar{Z}_i \otimes \bar{Z}_j \quad (58)$$

not only does not take the encoded information outside the DFS  $\tilde{\mathcal{H}}_2(0)$  of each of the  $N$  encoded qubits, i.e., satisfies the DFS preservation condition Eq. (33), it is also sufficient to generate a universal set of logical gates over the logical DFS qubits. Moreover, this Hamiltonian is composed entirely of one- and two-body physical qubit operators, so it is physically implementable.

#### IV. COLLECTIVE DECOHERENCE AND DECOHERENCE FREE SUBSPACES

The collective dephasing model can be readily modified to give the more general collective decoherence model. The interaction Hamiltonian has the following form:

$$H_{SB} = \sum_{i,k} [g_{i,k}^z \sigma_i^z \otimes (b_k + b_k^\dagger) + g_{i,k}^+ \sigma_i^+ \otimes b_k + g_{i,k}^- \sigma_i^- \otimes b_k^\dagger] \quad (59)$$

where

$$\sigma^\pm = \frac{1}{2} (\sigma_x \mp i\sigma_y) \quad (60)$$

corresponds to the raising (+) and lowering (−) operators for the spin of a particle respectively, i.e.,

$$\sigma^+ |0\rangle = |1\rangle \quad (61a)$$

$$\sigma^+ |1\rangle = 0 \quad (61b)$$

$$\sigma^- |0\rangle = 0 \quad (61c)$$

$$\sigma^- |1\rangle = |0\rangle \quad (61d)$$

where 0 here corresponds to the null vector and should not be confused with the  $|0\rangle$  state. Thus, e.g.,  $\sigma^+ = |1\rangle\langle 0|$ . It should be noted that the first term in the summation of (59) corresponds to an energy conserving (dephasing) term while the second and third terms correspond to energy exchange via, respectively, phonon absorption/spin excitation, and spin relaxation/phonon emission.

By assuming that all qubits are coupled to the same bath (thereby invoking permutation symmetry):

$$g_{ik}^\alpha = g_k^\alpha, \quad \forall k, \quad \alpha \in \{+, -, z\} \quad (62a)$$

$$\Rightarrow H_{SB} = \sum_i \sigma_i^+ \otimes \overbrace{\sum_k g_k^+ b_k}^{B_+} + \sum_i \sigma_i^- \otimes \overbrace{\sum_k g_k^- b_k^\dagger}^{B_-} + \sum_i \sigma_i^z \otimes \overbrace{\sum_k g_k^z (b_k + b_k^\dagger)}^{B_z} \quad (62b)$$

$$= \sum_{\alpha \in \{+, -, z\}} S_\alpha \otimes B_\alpha, \quad (62c)$$

where

$$S_\alpha = \sum_{i=1}^N \sigma_i^\alpha \quad (63)$$

is the total spin operator acting on the entire system of  $N$  physical qubits. We can derive the following relations directly from the commutation relations of the Pauli matrices:

$$\left. \begin{aligned} [S_\pm, S_z] &= \pm 2S_\pm \\ [S_-, S_+] &= S_z \end{aligned} \right\} \text{commutation relations for SL(2) triple} \quad (64)$$

where  $SL(2)$  is a Lie algebra [4].

We wish to define the total angular momentum operator  $\vec{S}^2$  in terms of the angular momenta operators around each axis. It will be convenient to define the vector of angular momenta:  $\vec{S} = (S_x, S_y, S_z)$  where  $S_x \equiv S_+ + S_-$  and  $S_y \equiv i(S_+ - S_-)$ . We note

that  $\vec{S}^2 \equiv \vec{S} \cdot \vec{S} = \sum_{\alpha \in \{x, y, z\}} S_\alpha^2$  satisfies  $[\vec{S}^2, S_z] = 0$ . Since  $\vec{S}^2$  and  $S_z$  commute and are both Hermitian, they are simultaneously diagonalizable, i.e., they share a common orthonormal eigenbasis.

Recalling some basic results from the quantum theory of angular momenta, we note that for the basis  $\{|S, m_S\rangle\}$  where  $S$  represents the total spin quantum number of a set of spin-1/2 particles and  $m_S$  represents the total spin projection quantum number onto the  $z$ -axis, we can show that

$$\vec{S}^2 |S, m_S\rangle \equiv S(S+1) |S, m_S\rangle \quad (65a)$$

$$S_z |S, m_S\rangle \equiv m_S |S, m_S\rangle \quad (65b)$$

where

$$S \in \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \frac{N}{2}\right\} \quad (66)$$

and

$$m_S \in \{-S, -S+1, \dots, S-1, S\}. \quad (67)$$

Keeping in mind that the basis states of the good subspace are eigenvectors of the interaction Hamiltonian and also satisfy Eq. (32) for  $\alpha \in \{+, -, z\}$ , let us examine the cases  $N = 1, 2, 3$  and 4 in turn.

### A. One Physical Qubit

For a single physical qubit ( $N = 1$ ), the basis  $\{|0\rangle, |1\rangle\}$  corresponds to that of our familiar spin- $\frac{1}{2}$  particle, with  $S = \frac{1}{2}$  and  $m_S = \pm \frac{1}{2}$ . We identify our logical zero and one states as follows

$$|0\rangle = |S = \frac{1}{2}, m_S = \frac{1}{2}\rangle \quad (68a)$$

$$|1\rangle = |S = \frac{1}{2}, m_S = -\frac{1}{2}\rangle \quad (68b)$$

### B. Two Physical Qubits

For two physical qubits ( $N = 2$ ), which we label  $A$  and  $B$ , with individual spins  $S_A = \frac{1}{2}$  and  $S_B = \frac{1}{2}$ , we first note that the prescription for adding angular momentum (or spin) given  $\vec{S}_A$  and  $\vec{S}_B$ , is to form the new spin operator  $\vec{S} = \vec{S}_A + \vec{S}_B$  with eigenvalues

$$S \in \{|S_A - S_B|, \dots, S_A + S_B\} \quad (69)$$

with the corresponding spin projection eigenvalues

$$m_S \in \{-S, \dots, S\}. \quad (70)$$

Thus, for two physical qubits, we see that the total spin eigenvalues  $S_{(N=2)}$  can only take the value 0 or 1. For  $S_{(N=2)} = 0$ , we see that  $m_S$  can only take the value 0 (singlet subspace) whereas when  $S_{(N=2)} = 1$ ,  $m_S$  can take any one of the three values  $-1, 0, 1$  (triplet subspace). For our singlet subspace,

$$|S_{(N=2)} = 0, m_S = 0\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \quad (71)$$

we see that  $S_z |S_{(N=2)} = 0, m_S = 0\rangle = 0$  for our system operator  $S_z = \sigma_1^z \otimes I + I \otimes \sigma_2^z$ . In fact,  $S_\alpha |S_{(N=2)} = 0, m_S = 0\rangle = 0$  for  $\alpha \in \{+, -, z\}$  where  $S_\alpha = \sum_i \sigma_i^\alpha$ . Similarly, it can also be shown that  $\vec{S}^2 |S_{(N=2)} = 0, m_S = 0\rangle = 0$ . Since the singlet state clearly satisfies condition (32), we conclude that  $|S_{(N=2)} = 0, m_S = 0\rangle$  is by itself a one-dimensional DFS. However, we also note that the triplet states are not eigenstates of  $S_z, S_+, S_-$  and thus violate Eq. (32).

### C. Three Physical Qubits

For three physical qubits ( $N = 3$ ), let us label the physical qubits  $A$ ,  $B$  and  $C$  each with corresponding total spins  $S_A = \frac{1}{2}$ ,  $S_B = \frac{1}{2}$  and  $S_C = \frac{1}{2}$ . If we think of this system as a combination of a pair of spins ( $A$  and  $B$ ) with another spin  $C$ , we can again apply our rule for adding angular momenta which gives us from combining our pair of physical qubits into a  $S_{(N=2)} = 0$  system with a spin- $\frac{1}{2}$  particle, eigenvalues of the total spin operator of

$$S_{(N=3)} = \left| 0 - \frac{1}{2} \right|, \dots, \left| 0 + \frac{1}{2} \right| = \frac{1}{2}$$

with corresponding spin projection eigenvalues  $m_S = \pm \frac{1}{2}$ . If instead we chose to combine our pair of physical qubits  $A$  and  $B$  into a  $S_{(N=2)} = 1$  system with a spin- $\frac{1}{2}$  particle, the eigenvalues of the total spin operator would be

$$S_{(N=3)} = \left| 1 - \frac{1}{2} \right|, \dots, \left| 1 + \frac{1}{2} \right| = \frac{1}{2}, \frac{3}{2}$$

with corresponding spin projection eigenvalues  $m_S = \pm \frac{1}{2}$  for  $S_{(N=3)} = \frac{1}{2}$  or  $m_S = \pm \frac{1}{2}, \pm \frac{3}{2}$  for  $S_{(N=3)} = \frac{3}{2}$ . These distinct cases arise because there are 2 distinct ways we can get a total spin of  $S = \frac{1}{2}$  from a system with 3 physical qubits, either with two of the qubits combined as a spin-1 system and then combined with the spin- $\frac{1}{2}$  particle or alternatively with two qubits combined as a spin-0 system and subsequently combined with the remaining spin- $\frac{1}{2}$  particle.

### D. Generalization to $N$ physical qubits

The extension of this idea of combining spin angular momenta is straightforward. There is an inductive method of building up from the above procedure to higher  $N$ . Suppose we wish to build up the spin states of  $N$  physical qubits. We would first build up the states for a set of  $N - 1$  physical qubits and then couple the spin of the last qubit.

Suppose we consider the case with  $N = 4$  physical qubits. We can create a Bratelli diagram for this scenario (Figure 2). The decoherence free states lie on the axis where  $S = 0$ . There are two possible paths to build up the states from  $N = 0$  to  $N = 4$ . So we can construct a qubit with each logical state  $|0_L\rangle$  and  $|1_L\rangle$  equal to a decoherence free state indexed by the path label  $\lambda$ .

$$|0_L\rangle = \text{[Diagram: A blue path with 4 nodes, starting at level 0, going up to level 1, down to level 2, up to level 3, and down to level 4.]} = |S = 0, m_S = 0, \lambda = 0\rangle \quad (72)$$

$$|1_L\rangle = \text{[Diagram: A red path with 4 nodes, starting at level 0, going up to level 1, up to level 2, down to level 3, and down to level 4.]} = |S = 0, m_S = 0, \lambda = 1\rangle \quad (73)$$

Before proceeding, let us establish some notation. Define the singlet state  $|s\rangle_{ij}$  over the  $i^{\text{th}}$  and  $j^{\text{th}}$  qubit as

$$|s\rangle_{ij} \equiv \frac{1}{\sqrt{2}} (|0_i 1_j\rangle - |1_i 0_j\rangle) \quad (74)$$

and the three triplet states as

$$|t^-\rangle_{ij} \equiv |1_i 1_j\rangle = |S = 1, m_S = -1\rangle \quad (75a)$$

$$|t^0\rangle_{ij} \equiv \frac{1}{\sqrt{2}} (|0_i 1_j\rangle + |1_i 0_j\rangle) = |S = 1, m_S = 0\rangle \quad (75b)$$

$$|t^+\rangle_{ij} \equiv |0_i 0_j\rangle = |S = 1, m_S = 1\rangle \quad (75c)$$

For  $N = 4$  physical qubits, the logical zero  $|0_L\rangle$  is given by

$$|0_L\rangle = \text{[Diagram: A blue path with 4 nodes, starting at level 0, going up to level 1, down to level 2, up to level 3, and down to level 4.]} \quad (76a)$$

$$= |\text{singlet}\rangle \otimes |\text{singlet}\rangle \quad (76b)$$

$$= |s\rangle_{12} \otimes |s\rangle_{34} \quad (76c)$$

On the other hand the logical one  $|1_L\rangle$  will be later seen to be given by

$$|1_L\rangle = \text{[Diagram: A red path with 4 nodes, starting at level 0, going up to level 1, up to level 2, down to level 3, and down to level 4.]} = \frac{1}{\sqrt{3}} [ |t^+\rangle_{12} \otimes |t^-\rangle_{34} + |t^-\rangle_{12} \otimes |t^+\rangle_{34} - |t^0\rangle_{12} \otimes |t^0\rangle_{34} ] \quad (77)$$

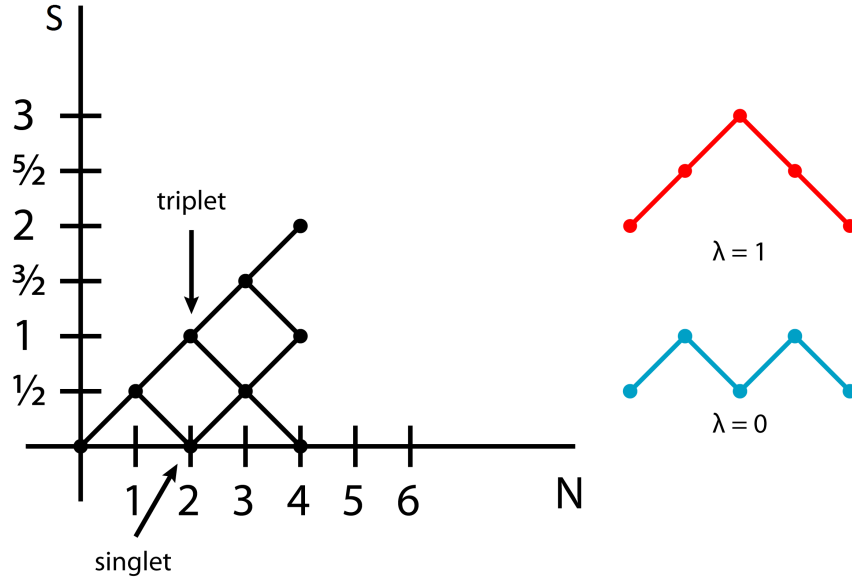


FIG. 2. Example Bratelli diagram for  $N = 4$  physical qubits. The decoherence free states lie on the points of the axis where  $S = 0$ . There are two ways of getting to  $S = 0$  when  $N = 4$  because there are two possible paths starting from  $N = 0$ . So we can realize a qubit by setting the logical computational basis states  $|0_L\rangle$  and  $|1_L\rangle$  to these two decoherence free states. Parameter  $\lambda$  indexes the two possible paths for  $N = 4$  physical qubits.

In a similar fashion, for  $N = 6$  physical qubits, we have for the logical zero

$$|0_L\rangle = |s\rangle_{12} \otimes |s\rangle_{34} \otimes |s\rangle_{56} \quad (78)$$

Note that permutations of qubit labels are permissible and can be used to define alternative basis states. Actually, we shall see in the next section that such permutations can be used to implement logical operations on the logical qubits.

### E. Higher Dimensions and Encoding Rate

Clearly, more paths exist as  $N$  grows. We thus have potentially more logical states available as we increase  $N$  because the logical states correspond to the distinct paths leading to each decoherence free state. There exists a combinatorial formula for the number of paths to each point in the Bratelli diagram with  $S = 0$  for a given  $N$  and hence for the dimension  $d_N$  of the DFS  $\tilde{\mathcal{H}}(N)$  of  $N$  spin- $\frac{1}{2}$  physical qubits

$$d_N \equiv \dim(\text{DFS}(N)) = \frac{N!}{(N/2)!(N/2+1)!} \quad (79)$$

As in the case of collective dephasing [Eq. (54)] we can determine the encoding rate from the above formula. The encoding rate  $r_N$  is the number of logical qubits  $N_L$  we obtain divided by the number of physical qubits  $N$  we put into the system. We can construct logical qubits from the logical states in the decoherence free subspace  $\tilde{\mathcal{H}}(N)$ , and the number  $N_L$  of logical qubits is logarithmic in the number of logical states of  $\tilde{\mathcal{H}}(N)$  with  $N_L = \log_2(d_N)$ . So the encoding rate  $r_N$  is

$$r_N \equiv \frac{\# \text{ of DFS qubits in } \tilde{\mathcal{H}}(N)}{\# \text{ of physical qubits}} = \frac{N_L}{N} = \frac{\log_2 d_N}{N}. \quad (80)$$

It can be shown using Stirling's approximation

$$\log_2 N! \approx (N + 1/2) \log_2 N - N \quad (81)$$

for  $N \gg 1$ , that the rate

$$r_N \approx 1 - \frac{3 \log_2 N}{2N} \quad (82)$$

for  $N \gg 1$  and hence that the rate  $r_N$  asymptotically approaches unity

$$\lim_{N \rightarrow \infty} r_N = 1. \quad (83)$$

This implies that when  $N$  is very large, remarkably we get about as many logical qubits out of our system as physical qubits we put into the system.

### F. Logical Operations on the DFS of Four Qubits

Consider once more the collective decoherence model. As we saw in Eq. (62) the system-bath Hamiltonian has the form:

$$H_{SB} = \sum_{\alpha=+, -, z} S_{\alpha} \otimes B_{\alpha} \quad (84)$$

where

$$S_{\alpha} = \sum_{i=1}^N \sigma_i^{\alpha}. \quad (85)$$

Recall that for  $N = 4$ , we have two DF states with  $S = 0$ . When we work out the states in the computational basis we find from Eqs. (76) and (77) that the corresponding logical qubits for this DFS are:

$$|\bar{0}\rangle = \frac{1}{2} (|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle) \quad (86a)$$

$$|\bar{1}\rangle = \frac{1}{\sqrt{3}} \left( |1100\rangle + |0011\rangle - \frac{1}{2}|0101\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle - \frac{1}{2}|1010\rangle \right) \quad (86b)$$

Suppose that we have a string of information qubits that we wish to protect. One way we can achieve this goal is to implement a DFS (an error avoiding code). For our case, this is to group the qubits into blocks of length of 4, and encode each block into the logical qubits stated above.

Now,  $\forall x, y \in \{0, 1\}$ , define the exchange operation  $E_{ij}$  on the state  $|x\rangle_i \otimes |y\rangle_j$  to be:

$$E_{ij} (|x\rangle_i \otimes |y\rangle_j) \equiv |y\rangle_i \otimes |x\rangle_j \quad (87)$$

Thus,  $E_{ij}$  has the following matrix representation in the standard basis of two qubits:

$$E_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (88)$$

and it is trivial that

$$[H_{SB}, E_{ij}] = 0 \quad \forall i, j \quad (89)$$

In the following we are going to see that  $E_{ij}$ 's can be used to generate the logical  $X, Y, Z$  operators for the logical qubits in our case.

Consider the operator  $(-E_{12})$ , and observe its operation on the logical qubits:

$$(-E_{12}) |\bar{0}\rangle = -(E_{12}) \frac{1}{2} (|0101\rangle - |0110\rangle - |1001\rangle + |1010\rangle) \quad (90a)$$

$$= -\frac{1}{2} (|1001\rangle - |1010\rangle - |0101\rangle + |0110\rangle) \quad (90b)$$

$$= |\bar{0}\rangle \quad (90c)$$

$$(-E_{12}) |\bar{1}\rangle = -(E_{12}) \frac{1}{\sqrt{3}} \left( |1100\rangle + |0011\rangle - \frac{1}{2}|0101\rangle - \frac{1}{2}|0110\rangle - \frac{1}{2}|1001\rangle - \frac{1}{2}|1010\rangle \right) \quad (90d)$$

$$= -\frac{1}{\sqrt{3}} \left( |1100\rangle + |0011\rangle - \frac{1}{2}|1001\rangle - \frac{1}{2}|1010\rangle - \frac{1}{2}|0101\rangle - \frac{1}{2}|0110\rangle \right) \quad (90e)$$

$$= -|\bar{1}\rangle \quad (90f)$$

We can see that  $(-E_{12})$  acts like a  $Z$  operator on the logical qubits.

Similarly, we may check that  $\frac{1}{\sqrt{3}}(E_{23} - E_{13})$  acts like an  $X$  operator on the logical qubits.

Thus, we may define one set of  $X, Y, Z$  operators for the DFS in our case to be:

$$\bar{\sigma}^z \equiv (-E_{12}) \quad (91a)$$

$$\bar{\sigma}^x \equiv \frac{1}{\sqrt{3}}(E_{23} - E_{13}) \quad (91b)$$

$$\bar{\sigma}^y \equiv \frac{i}{2}[\bar{\sigma}^x, \bar{\sigma}^z] \quad (91c)$$

As we saw in Section III C, with the  $\bar{\sigma}^x$  and  $\bar{\sigma}^z$  operations we can construct arbitrary qubit rotations via the Euler angle formula:

$$\exp(i\theta\hat{n} \cdot \vec{\bar{\sigma}}) = \exp(i\alpha\bar{\sigma}^x) \exp(i\beta\bar{\sigma}^z) \exp(i\gamma\bar{\sigma}^x) \quad (92)$$

To perform universal quantum computation we also need to construct entangling logical operations between the logical qubits. This too can be done using entirely using exchange operations. See [5] for the original construction of such a gate between the logical qubits of the 4-qubit DFS code.

## V. NOISELESS/DECOHERENCE FREE SUBSYSTEMS

### A. Representation theory of matrix algebras

We begin this section by stating a theorem in representation theory of matrix algebras.

Recall the general form of the system-bath Hamiltonian,  $H_{SB} = \sum_{\alpha} S_{\alpha} \otimes B_{\alpha}$ . Let  $\mathcal{A} = \{\text{poly}(S_{\alpha})\}$  be the associative algebra generated by all the system operators  $S_{\alpha}$  (all sums and products of such operators). The system Hilbert space is  $\mathcal{H}_S = (\mathbb{C}^2)^{\otimes N} = \mathbb{C}^{2^N}$ .

**Theorem 2.** Assume that  $\mathcal{A}$  is  $\dagger$ -closed (i.e.  $a \in \mathcal{A} \Rightarrow a^{\dagger} \in \mathcal{A}$ ) and that  $I \in \mathcal{A}$ . Then

$$\mathcal{A} \cong \bigoplus_j I_{n_j \times n_j} \otimes \mathcal{M}_{d_j \times d_j}(\mathbb{C}). \quad (93)$$

Here  $\mathcal{M}_{d \times d}(\mathbb{C})$  denotes the algebra of complex-valued  $d \times d$  matrices  $\{M_{d \times d}(\mathbb{C})\}$ , while as usual  $I$  is the identity matrix. The number  $j$  is the label of an irreducible representation (irrep) of  $\mathcal{A}$ ,  $n_j$  is the degeneracy of the  $j$ th irrep, and  $d_j$  is the dimension of the  $j$ th irrep.

From Theorem 2, we know that every  $S_{\alpha}$  has a matrix representation in the form  $S_{\alpha} = \bigoplus_j I_{n_j \times n_j} \otimes M_{d_j \times d_j}$ :

$$S_{\alpha} = \begin{bmatrix} J_1 & 0 & 0 & \cdots \\ 0 & J_2 & 0 & \cdots \\ 0 & 0 & J_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (94)$$

where  $J_j = I_{n_j \times n_j} \otimes M_{d_j \times d_j}$ :

$$J_j = \begin{bmatrix} M_{d_j \times d_j} & 0 & 0 & \cdots \\ 0 & M_{d_j \times d_j} & 0 & \cdots \\ 0 & 0 & M_{d_j \times d_j} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (95)$$

Therefore, we have the following Corollary resulting from the previous Theorem:

**Corollary 1** ([6]). *The system Hilbert space can be decomposed as*

$$\mathcal{H}_S = \bigoplus_j \mathbb{C}^{n_j} \otimes \mathbb{C}^{d_j}, \quad (96)$$

*and the subsystem factors  $\mathbb{C}^{n_j}$ 's are unaffected by decoherence.*

The above result can be easily seen since the system operators  $S_\alpha = \bigoplus_J I_{n_J \times n_J} \otimes M_{d_J \times d_J}$  act as the identity operator (hence do nothing) on each of the terms  $\mathbb{C}^{n_j}$ . This result is important since we now have decoherence-free subsystems, which we can use to store protected quantum information.

Recall that the “error algebra”  $\mathcal{A} = \{\text{poly}(S_\alpha)\}$  acts trivially on the components  $\mathbb{C}^{n_J}$  in Eq. (96). Thus we can noiselessly store quantum information in one of the components  $\mathbb{C}^{n_J}$ . These components  $\mathbb{C}^{n_J}$  are called *noiseless subsystems* (NS). Note that the dimension of  $\mathcal{H}_S = (\mathbb{C}^2)^{\otimes N} = \mathbb{C}^{2^N}$  can be decomposed as

$$2^N = \sum_J n_J d_J. \quad (97)$$

Define the commutant  $\mathcal{A}'$  of  $\mathcal{A}$  to be the set

$$\mathcal{A}' = \{X : [X, A] = 0, \forall A \in \mathcal{A}\}. \quad (98)$$

This set also forms a  $\dagger$ -closed algebra and is reducible to, over the same basis as  $\mathcal{A}$ ,

$$\mathcal{A}' \cong \bigoplus_J \mathcal{M}_{n_J \times n_J}(\mathbb{C}) \otimes I_{d_J \times d_J}. \quad (99)$$

These are the logical operations for performing quantum computation: they act non-trivially on the noiseless subsystems  $\mathbb{C}^{n_J}$ .

## B. Example: collective decoherence revisited

### 1. General structure

Let's return to the collective decoherence model. Recall that collective decoherence on  $N$  qubits is characterized by the system operators  $S_\alpha = \sum_{i=1}^N \sigma_i^\alpha$ , for  $\alpha \in \{+, -, z\}$ . In this case, the system space is

$$\mathcal{H}_S = \bigoplus_{J=0(1/2)}^{N/2} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}, \quad (100)$$

where  $J$  labels the total spin, and the sum is from  $J = 0$  or  $J = 1/2$  if  $N$  is even or odd, respectively. For a fixed  $J$ , there are  $2J + 1$  different eigenvalues of  $m_J$ , and hence

$$d_J = 2J + 1. \quad (101)$$

By using angular momentum addition rules, one can prove that

$$n_J = \frac{(2J + 1)N!}{(N/2 + 1 + J)!(N/2 - J)!}, \quad (102)$$

which is equal to the number of paths from the origin to the vertex  $(N, J)$  on the Bratteli diagram. For example, when  $N = 3$ :  $n_{1/2} = \frac{2 \cdot 3!}{3!1!} = 2$ . We have

$$\mathcal{H}_S = \mathbb{C}^{n_0} \otimes \mathbb{C}^1 \oplus \mathbb{C}^{n_1} \otimes \mathbb{C}^3 \oplus \dots \quad (103)$$

for  $N$  even, and

$$\mathcal{H}_S = \mathbb{C}^{n_{1/2}} \otimes \mathbb{C}^2 \oplus \mathbb{C}^{n_{3/2}} \otimes \mathbb{C}^4 \oplus \dots, \quad (104)$$

for  $N$  odd. Note that when  $J = 0$  (so that  $d_J = 1$ ),  $\mathbb{C}^1$  is just a scalar and  $\mathbb{C}^{n_0} \otimes \mathbb{C}^1 = \mathbb{C}^{n_0}$ , in which a DFS arises.



The noiseless subsystems corresponding to different values of  $J$  for a given  $N$  can be computed by using the addition of angular momentum, as illustrated below.



## 2. The three qubit code for collective decoherence

The smallest  $N$  which encodes one qubit in a noiseless subsystem is  $N = 3$ . In this case,

$$\mathcal{H}_S^{(N=3)} = \mathbb{C}^2 \otimes \mathbb{C}^2 \oplus \mathbb{C}^1 \otimes \mathbb{C}^4, \quad (105)$$

Thus we can encode one qubit in the first factor  $\mathbb{C}^2$  of  $J = 1/2$ . The two paths of  $|\bar{0}\rangle$  and  $|\bar{1}\rangle$  are respectively  ( $\lambda = 0$ ) and  ( $\lambda = 1$ ). The end points of these two paths each have two spin projections  $m_J = \pm 1/2$  (since they correspond to a total spin  $J = 1/2$ ). Using the state notation  $|J, \lambda, m_J\rangle$ , we thus have

$$|\bar{0}\rangle = \alpha |1/2, 0, -1/2\rangle + \beta |1/2, 0, 1/2\rangle = |1/2, 0\rangle \otimes (\alpha |-1/2\rangle + \beta |1/2\rangle), \quad (106a)$$

$$|\bar{1}\rangle = \alpha |1/2, 1, -1/2\rangle + \beta |1/2, 1, 1/2\rangle = |1/2, 1\rangle \otimes (\alpha |-1/2\rangle + \beta |1/2\rangle), \quad (106b)$$

where  $\alpha$  and  $\beta$  are completely arbitrary. Or using the vector form, we have

$$|\bar{0}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad |\bar{1}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \quad (107)$$

Suppose we want to encode a state  $|\psi\rangle = a|0\rangle + b|1\rangle$ . The encoded state is

$$|\bar{\psi}\rangle = a|\bar{0}\rangle + b|\bar{1}\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (108)$$

where we only care about the encoded information  $a$  and  $b$ . Notice how this last result precisely corresponds to the  $\mathbb{C}^2 \otimes \mathbb{C}^2$  term in Eq. (105). Thus,  $\alpha$  and  $\beta$  are a “gauge freedom”; their precise values don’t matter.

The interaction Hamiltonian restricted to the system  $S$  is of the form

$$H_{SB}|_S = \bigoplus_{J=1/2}^{3/2} I_{n_J} \otimes \mathcal{M}_{d_J} \quad (109a)$$

$$= I_2 \otimes \mathcal{M}_2 \oplus I_1 \otimes \mathcal{M}_4 \quad (109b)$$

$$= \left[ \begin{array}{c|c} I_2 \otimes \mathcal{M}_2 & \\ \hline & \mathcal{M}_4 \end{array} \right]. \quad (109c)$$

What this means is that the term  $I_2 \otimes \mathcal{M}_2$  acts on  $|\bar{\psi}\rangle$  and leaves its first factor alone (this is good since that’s where we store the qubit), but applies some arbitrary matrix  $\mathcal{M}_2$  to the second factor (we don’t care).  $\mathcal{M}_4$  acts on the  $\mathbb{C}^1 \otimes \mathbb{C}^4$  subspace, where we don’t store any quantum information.

We can check that the dimensions satisfy Eq. (97):

$$\sum_{J=1/2}^{3/2} n_J d_J = n_{1/2} d_{1/2} + n_{3/2} d_{3/2} = 2 \cdot 2 + 1 \cdot 4 = 8 = 2^3. \quad (110)$$

Let’s now find explicit expressions for the basis state of the three-qubit noiseless subsystem. Recall that  $|0\rangle = |J = 1/2, m_J = 1/2\rangle$ ,  $|1\rangle = |1/2, -1/2\rangle$ , the singlet state  $|s\rangle = |0, 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$  and the triplet states are  $|t_+\rangle = |1, 1\rangle = |00\rangle$ ,  $|t_-\rangle = |1, -1\rangle = |11\rangle$ , and  $|t_0\rangle = |1, 0\rangle = \frac{1}{2}(|01\rangle + |10\rangle)$ . We now derive the four  $J = 1/2$  states by using the addition of angular momentum and

Clebsch-Gordan coefficients.

$$|1/2, 0, -1/2\rangle = |s\rangle \otimes |m_3 = -1/2\rangle \quad (111a)$$

$$= \frac{1}{\sqrt{2}}(|011\rangle - |101\rangle) \quad (111b)$$

$$|1/2, 0, 1/2\rangle = |s\rangle \otimes |0\rangle \quad (111c)$$

$$= \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle) \quad (111d)$$

$$|1/2, 1, -1/2\rangle = \frac{1}{\sqrt{3}}(\sqrt{2}|J_{12} = 1, m_{J_{12}} = -1\rangle \otimes |m_3 = 1/2\rangle - |J_{12} = 1, m_{J_{12}} = 0\rangle \otimes |m_3 = -1/2\rangle) \quad (111e)$$

$$= \frac{1}{\sqrt{6}}(2|110\rangle - |011\rangle - |101\rangle) \quad (111f)$$

$$|1/2, 1, 1/2\rangle = \frac{1}{\sqrt{3}}(|J_{12} = 1, m_{J_{12}} = 0\rangle \otimes |m_3 = 1/2\rangle - \sqrt{2}|J_{12} = 1, m_{J_{12}} = 1\rangle \otimes |m_3 = -1/2\rangle) \quad (111g)$$

$$= \frac{1}{\sqrt{6}}(|010\rangle + |100\rangle - 2|001\rangle). \quad (111h)$$

These are the basis states that appear in Eq. (106), so they complete the specification of the three-qubit code.

### 3. Computation over the three-qubit code

Consider the permutation operator  $E_{ij} = \frac{1}{2}(I + \vec{\sigma}_i \cdot \vec{\sigma}_j)$  such that  $E_{ij}|x\rangle_i|y\rangle_j = |y\rangle_i|x\rangle_j$  for  $x, y \in \{0, 1\}$ . We have

$$E_{12}|1/2, 0, -1/2\rangle = \frac{1}{\sqrt{2}}(-|011\rangle + |101\rangle) = -|1/2, 0, -1/2\rangle \quad (112a)$$

$$E_{12}|1/2, 0, 1/2\rangle = \frac{1}{\sqrt{2}}(|100\rangle - |010\rangle) = -|1/2, 0, 1/2\rangle \quad (112b)$$

$$E_{12}|1/2, 1, -1/2\rangle = \frac{1}{\sqrt{6}}(2|110\rangle - |011\rangle - |101\rangle) = |1/2, 1, -1/2\rangle \quad (112c)$$

$$E_{12}|1/2, 1, 1/2\rangle = \frac{1}{\sqrt{6}}(|010\rangle + |100\rangle - 2|001\rangle) = |1/2, 1, 1/2\rangle. \quad (112d)$$

Thus  $E_{12}$  works as a logical  $-\sigma^z$ , in the sense that

$$E_{12} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = -\sigma^z \otimes I = -\bar{\sigma}^z \quad (113)$$

in the ordered basis of the four  $J = 1/2$  states given in Eq. (112). Again, this agrees with the  $\mathbb{C}^2 \otimes \mathbb{C}^2$  structure of the Hilbert subspace where we store our qubit.

Similarly, one can easily verify that

$$\frac{1}{\sqrt{3}}(E_{13} - E_{23}) = \sigma^x \otimes I = \bar{\sigma}^x. \quad (114)$$

Then  $\bar{\sigma}^y$  can be obtained from

$$2i\bar{\sigma}^y = [\bar{\sigma}^z, \bar{\sigma}^x]. \quad (115)$$

### C. Intermediate Summary

To conclude, the idea of a DFS/NS is powerful: we can use naturally available symmetries to encode and hide quantum information, and we can compute over the encoded, hidden information. But often such symmetries are imperfect, and we need additional tools to protect quantum information. Such an approach, which adds active intervention to the passive DFS/NS approach, is dynamical decoupling.

## VI. DYNAMICAL DECOUPLING

As we saw in the discussion of noiseless subsystems, the error algebra  $\mathcal{A} = \{\text{poly}(S_\alpha)\}$  is isomorphic to a direct sum of copies of  $d_J \times d_J$  complex matrix algebras  $\mathcal{M}_{d_J \times d_J}(\mathbb{C})$ :

$$\mathcal{A} \cong \bigoplus_J I_{n_J \times n_J} \otimes \mathcal{M}_{d_J \times d_J}(\mathbb{C}), \quad (116)$$

where  $n_J$  is the degeneracy of the  $J$ th irrep and  $d_J$  is the dimension of the  $J$ th irrep. We can store quantum information in a factor  $\mathbb{C}^{n_J}$  when  $n_J > 1$ . However, from general principles (Noether's theorem) we know that degeneracy requires a symmetry, and in our case we would only have  $n_J > 1$  when the system-bath coupling has some symmetry. When there's no symmetry at all,  $n_J = 1$  for all  $J$ 's, and a DFS or NS may not exist. Starting in this section, we discuss how to “engineer” the system-bath coupling to have some symmetry.

### A. Decoupling single qubit pure dephasing

#### 1. The ideal pulse case

Consider a single qubit system with the pure dephasing system-bath coupling Hamiltonian

$$H_{SB} = \sigma^z \otimes B^z \quad (117)$$

and system Hamiltonian

$$H_S = \lambda(t) \sigma^x. \quad (118)$$

We assume that  $\lambda(t)$  is a fully controllable field, e.g., several pulses of a magnetic or electric field applied to the system. Assume these ideal pulses last for a period of time  $\delta$ , and with strength  $\lambda$ , and

$$\delta \lambda = \frac{\pi}{2}. \quad (119)$$

Assume that at  $t = 0$ , we turn on the pulse for a period of time  $\delta$ , then let the system and bath interact for a period of time  $\tau$ , and repeat this procedure, as shown in Fig. 3. In the *ideal* case,  $\delta \rightarrow 0$  and  $\lambda \rightarrow \infty$  while still satisfying  $\delta \lambda = \frac{\pi}{2}$ , which means the pulses are a series of delta functions. For simplicity we temporarily assume that  $H_B = 0$ .

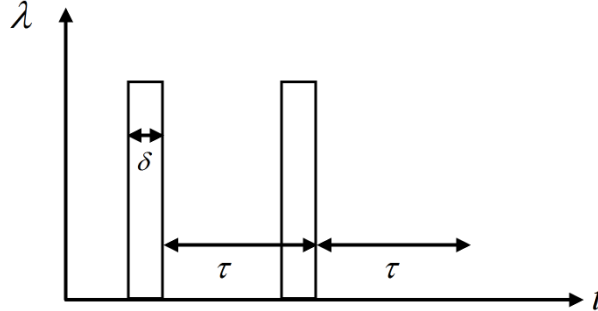


FIG. 3. Schematic of a dynamical decoupling pulse sequence. Pulses have width  $\delta$  and intervals of duration  $\tau$ . The modulation function  $\lambda(t)$  is responsible for switching the pulses on and off.

To formalize this “ideal pulse” scenario, let us define the system-bath “pulse-free” evolution operator  $f_\tau$ , and the unitary transformation caused by the pulse,  $X$ , as follows:

$$f_\tau \equiv e^{-i\tau H_{SB}}. \quad (120)$$

$$X \equiv e^{-i\frac{\pi}{2}\sigma^x} \otimes I_B = -ie^{-i\delta\lambda\sigma^x} \otimes I_B = -i\sigma^x \otimes I_B. \quad (121)$$

In the case of an ideal pulse ( $\delta \rightarrow 0, \lambda \rightarrow \infty$ ) there is no system-bath interaction during the time the pulse is turned on, since the duration of the pulse is 0. Then the joint system-bath evolution operator at time  $t = 2\tau$  is (dropping overall factors of  $i$  and minus signs)

$$X f_\tau X f_\tau = \sigma^x e^{-i\tau H_{SB}} \sigma^x e^{-i\tau H_{SB}} = e^{-i\tau \sigma^x H_{SB} \sigma^x} e^{-i\tau H_{SB}}. \quad (122)$$

where in the second equality we used the identity

$$Ue^A U^\dagger = e^{UAU^\dagger}, \quad (123)$$

valid for any operator  $A$  and unitary  $U$ .

On the other hand, since the Pauli matrices are Hermitian and every pair of distinct Pauli matrices anticommutes,

$$\{\sigma_\alpha, \sigma_\beta\} = 0, \quad \alpha \neq \beta, \quad (124)$$

where the anti-commutator is defined as

$$\{A, B\} \equiv AB + BA \quad (125)$$

for any pair of operators  $A$  and  $B$ , it follows that the sign of  $H_{SB}$  is flipped:

$$\sigma^x H_{SB} \sigma^x = \sigma^x \sigma^z \sigma^x \otimes B^z \quad (126a)$$

$$= -\sigma^z \otimes B^z, \quad (126b)$$

$$= -H_{SB}. \quad (126c)$$

This means that the evolution under  $H_{SB}$  has been effectively time-reversed!

Indeed, if we now substitute Eq. (126c) into Eq. (122) we obtain

$$X f_\tau X f_\tau = e^{+i\tau H_{SB}} e^{-i\tau H_{SB}} = I. \quad (127)$$

Thus, the bath has no effect on the system at the instant  $t = 2\tau$ . In other words, for a fleeting instant, at  $t = 2\tau$ , the system is completely decoupled from the bath. Clearly, if we were to repeat Eq. (122) over and over, the system would “stroboscopically” decouple from the bath every  $2\tau$ .

## 2. The real pulse case

Unfortunately, in the real world, pulses cannot be described by  $\delta$  functions, because that would require infinite energy. Generally, the pulse must be described by some continuous function  $\lambda(t)$  in the time domain, which may or may not be a pulse (see, e.g., Ref. [7], for a variational solution of the optimal modulation for the pure dephasing problem). Then, during the period when the pulse is applied to the system, the system-bath Hamiltonian cannot be neglected, so we must take it into account. Keeping the assumption  $H_B = 0$  for the time being, we have to modify the pulse to

$$X = e^{-i\delta(\lambda\sigma^x + H_{SB})}. \quad (128)$$

If  $\lambda \gg \|H_{SB}\|$  and  $\delta\lambda = \pi/2$  (we’ll define the norm momentarily), then it’s true that  $X \approx \sigma^x \otimes I_B$ , i.e., we can approximate the ideal pulse case of Eq. (121). Let us now see how good of an approximation this is.

To deal with the real pulse case, we first recall the Baker-Campbell-Hausdorff (BCH) formula (see any advanced book on matrices, e.g., [8]):

$$e^{\epsilon(A+B)} = e^{\epsilon A} e^{\epsilon B} e^{(\epsilon^2/2)[A,B] + \mathcal{O}(\epsilon^3)}, \quad (129)$$

for any pair of operators  $A$  and  $B$ . Now, set  $\epsilon = -i\delta$ ,  $A = \lambda\sigma^x$ ,  $B = H_{SB} = \sigma^z \otimes B^z$ . Then the real pulse is

$$X = \underbrace{e^{-i\delta\lambda\sigma^x}}_{\text{ideal pulse}} \underbrace{e^{-i\delta H_{SB}}}_{\text{OK}} \underbrace{e^{-\delta^2\lambda[\sigma^x, H_{SB}]/2 + \mathcal{O}(\delta^3)}}_{\text{does damage}} \quad (130)$$

The first exponential term is just the ideal pulse, and the second is OK as well (we will see that shortly), but the third term will cause the pulse sequence to operate imperfectly. Let’s analyze the pulse sequence subject to this structure of the real pulse.

First, let us define the operator norm:

$$\|A\| \equiv \sup_{|\psi\rangle} \frac{\sqrt{\langle\psi|A^\dagger A|\psi\rangle}}{\langle\psi|\psi\rangle}, \quad (131)$$

which reduces to the absolute value of the largest eigenvalue of  $A$  when  $A$  is Hermitian. The operator norm is an example of a unitarily invariant (ui) norm: If  $U$  and  $V$  are unitary, and  $A$  is some operator, a norm is said to be unitarily invariant if

$$\|UAV\|_{\text{ui}} = \|A\|_{\text{ui}}. \quad (132)$$

Such norms are submultiplicative over products and distributive over tensor products [9]:

$$\|AB\|_{\text{ui}} \leq \|A\|_{\text{ui}} \|B\|_{\text{ui}}, \quad \|A \otimes B\|_{\text{ui}} = \|A\|_{\text{ui}} \|B\|_{\text{ui}}. \quad (133)$$

Then, using  $\|\sigma^x\| = 1$  (the eigenvalues of  $\sigma_x$  are  $\pm 1$ ), we have

$$\|H_{SB}\| = \|\sigma^x \otimes B^z\| = \|B^z\|. \quad (134)$$

Using this we find

$$\|\delta^2 \lambda [\sigma^x, H_{SB}]/2\| \leq \delta^2 \lambda (\|\sigma^x H_{SB}\| + \|H_{SB} \sigma^x\|)/2 \quad (135a)$$

$$\leq (\pi/2) \delta \|\sigma^x\| \|H_{SB}\| \quad (135b)$$

$$= \mathcal{O}(\delta \|B^z\|), \quad (135c)$$

where we used the triangle inequality and  $\delta \lambda = \pi/2$ . So, we arrive at the important conclusion that the pulse width should be small compared to the inverse of the system-bath coupling strength, i.e.,  $\delta \ll 1/\|B^z\|$ , should be satisfied assuming  $\|B^z\|$  is finite. This assumption won't always be satisfied (e.g., it does not hold for the spin-boson model), in which case different analysis techniques are required. In particular, operators norms will have to be replaced by correlation functions, which remain finite even when operator norms are formally infinite (see, e.g., [10]). But, for now we shall simply assume that all operators norms we shall encounter are indeed finite.

Let's Taylor expand the "damage" term to lowest order:

$$e^{-\delta^2 \lambda [\sigma^x, H_{SB}]/2} = I - \delta^2 \lambda [\sigma^x, H_{SB}]/2 + \mathcal{O}(\delta^3) \quad (136a)$$

$$= I + \mathcal{O}(\delta \|B^z\|). \quad (136b)$$

Putting everything together, including  $e^{-i\delta \lambda \sigma^x} = -i\sigma^x$ , the evolution subject to the real pulse is, from Eq. (130) (again dropping overall phase factors)

$$X f_\tau X f_\tau = [\sigma^x e^{-i\delta H_{SB}} (I + \mathcal{O}(\delta \|B^z\|))] e^{-i\tau H_{SB}} [\sigma^x e^{-i\delta H_{SB}} (I + \mathcal{O}(\delta \|B^z\|))] e^{-i\tau H_{SB}} \quad (137a)$$

$$= e^{-i(\tau+\delta)\sigma^x H_{SB} \sigma^x} e^{-i(\tau+\delta)H_{SB}} + \mathcal{O}(\delta \|B^z\|) + \mathcal{O}(\delta^2 \|B^z\|^2) \quad (137b)$$

$$= e^{i(\tau+\delta)H_{SB}} e^{-i(\tau+\delta)H_{SB}} + \mathcal{O}(\delta \|B^z\|) \quad (137c)$$

$$= I + \mathcal{O}(\delta \|B^z\|), \quad (137d)$$

so we see that the real pulse sequence has a first order pulse width correction.

Now let us recall that in fact  $H_B \neq 0$ . How does this impact the analysis? Both the free evolution and the pulse actually include  $H_B$ :

$$f_\tau = e^{-i\tau(H_{SB}+H_B)}, \quad (138a)$$

$$X = e^{-i\delta(\lambda\sigma^x+H_{SB}+H_B)}, \quad (138b)$$

so we need  $\lambda \gg \|H_{SB} + H_B\|$ . Set  $H'_{SB} = H_B + H_{SB}$ , and note that the ideal pulse commutes with  $H_B$ , so that

$$\sigma^x (H_{SB} + H_B) \sigma^x = -H_{SB} + H_B. \quad (139)$$

Substituting Eqs. (138a) and (138b) into Eq. (137) we then have:

$$X f_\tau X f_\tau = e^{-i(\tau+\delta)\sigma^x H'_{SB} \sigma^x} e^{-i(\tau+\delta)H'_{SB}} + \mathcal{O}(\delta \|H'_{SB}\|) \quad (140a)$$

$$= e^{-i(\tau+\delta)(-H_{SB}+H_B)} e^{-i(\tau+\delta)(H_{SB}+H_B)} + \mathcal{O}(\delta \|H'_{SB}\|) \quad (140b)$$

Setting  $A = H_{SB} + H_B$ ,  $B = -H_{SB} + H_B$ , and using the BCH formula (129) again, we have  $A + B = 2H_B$  and  $\|[A, B]\|/2 \leq \|H_B - H_{SB}\| \|H_B + H_{SB}\| \leq (\|H_{SB}\| + \|H_B\|)^2$ , so that Eq. (140) reduces to

$$X f_\tau X f_\tau = I_S \otimes e^{-2i(\tau+\delta)H_B} + \mathcal{O}[(\tau+\delta)^2 (\|H_{SB}\| + \|H_B\|)^2] + \mathcal{O}[\delta (\|H_{SB}\| + \|H_B\|)]. \quad (141)$$

Assuming that the pulses are very narrow, i.e.,  $\delta \ll \tau$  (recall that we anyhow need this for ideal pulses), we can neglect  $\delta$  relative to  $\tau$  in the second term, and so the smallness conditions are

$$\delta \ll \tau \ll 1/(\|B^z\| + \|H_B\|), \quad (142)$$

which replaces the earlier  $\delta \ll 1/\|B^z\|$  condition we derived when we ignored  $H_B$ .

### B. Decoupling single qubit general decoherence

Let us now consider the most general 1-qubit system-bath coupling Hamiltonian

$$H_{SB} = \sum_{\alpha=x,y,z} \sigma^\alpha \otimes B^\alpha. \quad (143)$$

Using the anticommutation condition Eq. (124) we have

$$\sigma^x H_{SB} \sigma^x = \sigma^x \otimes B^x - \sigma^y \otimes B^y - \sigma^z \otimes B^z, \quad (144)$$

so that the  $X f_\tau X f_\tau$  pulse sequence should cancel both the  $y$  and  $z$  contributions. The remaining problem is how to deal with the  $\sigma^x$  term in  $H_{SB}$ .

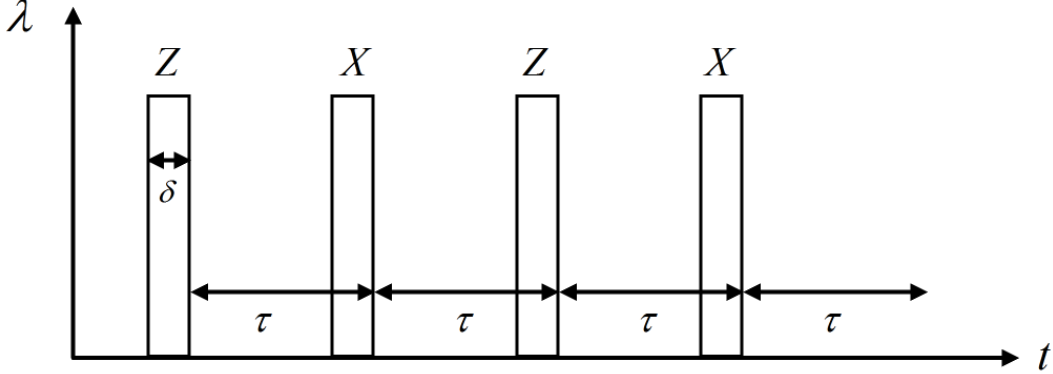


FIG. 4. Schematic of the pulse sequence used to suppress general single-qubit decoherence. This pulse sequence is sometimes called XY-4, or the universal decoupling sequence.

Let us assume that the pulses are ideal ( $\delta = 0$ ). We can remove the remaining  $\sigma^x$  term by inserting the sequence for pure dephasing into a second pulse sequence, designed to remove the  $\sigma_x$  term. This kind of recursive construction is very powerful, and we will see it again in Section IX.

Let the free evolution again be

$$f_\tau = e^{-i\tau H_{SB}}. \quad (145)$$

Then, after applying an  $X$ -type sequence,

$$X f'_{2\tau} \equiv f_\tau X f_\tau = e^{-i2\tau(\sigma^x \otimes B^x + H_B)} + \mathcal{O}(\tau^2). \quad (146)$$

To remove the remaining  $\sigma^x \otimes B^x$  we can apply a  $Y$ -type sequence to  $f'_{2\tau}$ :

$$f''_{4\tau} = Y f'_{2\tau} Y f'_{2\tau} \quad (147a)$$

$$= Y X f_\tau X f_\tau Y X f_\tau X f_\tau \quad (147b)$$

$$= Z f_\tau X f_\tau Z f_\tau X f_\tau. \quad (147c)$$

where as usual we dropped overall phase factors. Clearly,

$$f''_{4\tau} = e^{-i4\tau H_B} + \mathcal{O}(\tau^2), \quad (148)$$

so that at  $t = 4\tau$  the system is completely decoupled from the bath. This pulse sequence is shown in Fig. 4, and is the “universal decoupling sequence” (for a single qubit), since it removes a general system-bath interaction.

### VII. DYNAMICAL DECOUPLING AS SYMMETRIZATION

We saw in Eq. (147) that the sequence  $Z f_\tau X f_\tau Z f_\tau X f_\tau$  decouples a single qubit from an arbitrary bath (to first order). We constructed this sequence using a recursive scheme. In this section we would like to adopt a different perspective, which will help us generalize the theory beyond the single qubit case. This perspective is based on symmetrization [11].

Notice that, up to a global phase,

$$Z f_\tau X f_\tau Z f_\tau X f_\tau = (Z f_\tau Z) (Y f_\tau Y) (X f_\tau X) (I f_\tau I). \quad (149)$$

On the right hand side of (149) we see a clear structure: we are “cycling” over the group formed by the elements  $\{I, X, Y, Z\}$ . Note that because we are not concerned with global phases, this is not the Pauli group.<sup>1</sup> Rather, the four element group is the abelian Klein group, whose multiplication table is given by

$\times$	$I$	$X$	$Y$	$Z$
$I$	$I$	$X$	$Y$	$Z$
$X$	$X$	$I$	$Z$	$Y$
$Y$	$Y$	$Z$	$I$	$X$
$Z$	$Z$	$Y$	$X$	$I$

Returning to the decoupling discussion, to see why the sequence in Eq. (149) works, note that if we let  $A^\alpha = \sigma^\alpha \otimes B^\alpha$  we have

$$I f_\tau I = f_\tau = e^{-i\tau(A^x + A^y + A^z + H_B)} \quad (150)$$

$$X f_\tau X = e^{-i\tau\sigma^x H \sigma^x} = e^{-i\tau(A^x - A^y - A^z + H_B)} \quad (151)$$

$$Y f_\tau Y = e^{-i\tau\sigma^y H \sigma^y} = e^{-i\tau(-A^x + A^y - A^z + H_B)} \quad (152)$$

$$Z f_\tau Z = e^{-i\tau\sigma^z H \sigma^z} = e^{-i\tau(-A^x - A^y + A^z + H_B)} \quad (153)$$

Using the BCH expansion (129) again, we see that when we add all four of the exponents they cancel all  $A^\alpha$  terms perfectly, so that the right hand side of (149) is just

$$(Z f_\tau Z) (Y f_\tau Y) (X f_\tau X) (I f_\tau I) = e^{-4i\tau H_B} + \mathcal{O}(\tau^2), \quad (154)$$

just like in Eq. (148). This is the first order decoupling we were looking for.

From the right hand side of equation (149), we also gain some intuition as to what our strategy should be beyond the single qubit case. Again, define

$$f_\tau = \exp[-i\tau (H_{SB} + H_B)], \quad (155)$$

where now  $H_{SB}$  and  $H_B$  are completely general system-bath and pure-bath operators. Generalizing from Eq. (149), consider a group

$$\mathcal{G} = \{g_0, \dots, g_K\} \quad (156)$$

(with  $g_0 \equiv I$ ) of unitary transformations  $g_j$  acting purely on the system, and apply the following *symmetrization sequence*:

$$U(T) = \prod_{j=0}^K g_j^\dagger f_\tau g_j \quad (157a)$$

$$= \prod_{j=0}^K e^{-i\tau(g_j^\dagger H_{SB} g_j + H_B)} \quad (157b)$$

$$= e^{-iT\left(\frac{1}{K+1} \sum_{j=0}^K g_j^\dagger H_{SB} g_j + H_B\right)} + \mathcal{O}(T^2), \quad (157c)$$

where we used Eq. (123) in the second equality and the BCH formula in the third. The total time taken by the sequence is

$$T = (K + 1) \tau. \quad (158)$$

Now, define the effective, or average Hamiltonian

$$H'_{SB} = \frac{1}{K+1} \sum_{j=0}^K g_j^\dagger H_{SB} g_j. \quad (159)$$

Our strategy for general first order decoupling could be one of following:

<sup>1</sup> The Pauli group is the 16-element group  $\{\pm I, \pm X, \pm Y, \pm Z, \pm iI, \pm iX, \pm iY, \pm iZ\}$ .

1. Pick a group  $\mathcal{G}$  such that  $H'_{SB} = 0$ .
2. Pick a group  $\mathcal{G}$  such that  $H'_{SB} = I_S \otimes B'$ .

The first of these is precisely what we saw for decoupling a single qubit using the Pauli (or Klein) group, i.e., Eq. (154). To see when we can achieve the second strategy (which obviously included the first as a special case with  $B' = 0$ ), note that  $H'_{SB}$  belongs to the centralizer of the group  $\mathcal{G}$ , i.e.

$$H'_{SB} \in Z(\mathcal{G}) \equiv \{A | Ag = gA \ \forall g \in \mathcal{G}\}.$$

To prove this we only need to show that  $g^\dagger H'_{SB} g = H'_{SB}$  for all  $g \in \mathcal{G}$ , since this immediately implies that  $[H'_{SB}, g] = 0 \ \forall g \in \mathcal{G}$ . Indeed,

$$g^\dagger H_{SB} g = \frac{1}{K+1} \sum_{j=0}^K g^\dagger g_j^\dagger H_{SB} g_j g \quad (160a)$$

$$= \frac{1}{K+1} \sum_{j=0}^K (g_j g)^\dagger H_{SB} (g_j g) \quad (160b)$$

$$= H'_{SB}, \quad (160c)$$

since by group closure  $\{g_j g\}_{j=0}^K$  also covers all of  $\mathcal{G}$ .

The fact that  $H'_{SB}$  commutes with everything in  $\mathcal{G}$  means that we can apply Schur's Lemma [4]:

**Lemma 1** (Schur's Lemma). *Let  $\mathcal{G} = \{g_i\}$  be a group. Let  $T(\mathcal{G})$  be an irreducible  $d$ -dimensional representation of  $\mathcal{G}$  (i.e. not all of the  $T(g_i)$  are similar to a block-diagonal matrix). If there is a  $d \times d$  matrix  $A$  such that  $[A, g_i] = 0 \ \forall g_i \in \mathcal{G}$ , then  $A \propto I$ .*

Thus, it follows from this lemma that, provided we pick  $\mathcal{G}$  so that its matrix representation over the relevant system Hilbert space is irreducible, then indeed  $H'_{SB} \propto I_S$ .

For example,  $\mathcal{H}_S = (\mathbb{C}^2)^{\otimes n} = \mathbb{C}^{2^n}$  for  $n$  qubits; the dimension of the irrep should then be  $2^n$  in this case. Which decoupling group has a  $2^n$ -dimensional irrep over  $(\mathbb{C}^2)^{\otimes n}$ ? An example is the  $n$ -fold tensor product of the Pauli group:  $\mathcal{G} = \{I \otimes \dots \otimes I, X \otimes I \otimes \dots \otimes I, \dots, Z \otimes \dots \otimes Z\}$ . And indeed, this decoupling group suffices to decouple the most general system-bath Hamiltonian in the case of  $n$  qubits:

$$H_{SB} = \sum_{\alpha} \sigma_1^{\alpha_1} \otimes \dots \otimes \sigma_n^{\alpha_n} \otimes B^{\alpha}, \quad (161)$$

where  $\alpha = \{\alpha_1, \dots, \alpha_n\}$ , and  $\alpha_i \in \{0, x, y, z\}$ , with the convention that  $\sigma^0 = I$ . Fortunately, such a system-bath interaction is completely unrealistic, since it involves  $n+1$ -body interactions. “Fortunately,” since the decoupling group we just wrote down has  $K-1 = 4^n$  elements, so that the time it would take to apply just once symmetrization sequence (157a) grows exponentially with the number of qubits.

Actually, this approach using Schur's lemma is a bit too blunt. As we shall see, it is possible to drastically reduce the required resources for decoupling, for example by combining decoupling with DFS encoding, or by focusing on more reasonable models of system-bath interactions.

## VIII. COMBINING DYNAMICAL DECOUPLING WITH DFS

We saw that to decouple the general system-bath interaction  $H_{SB}$  in Eq. (161) would require a group with an exponentially large number of elements. This is not only impractical, it might also destroy any benefit we would hope to get from efficient quantum algorithms. Therefore we now consider ways to shorten the decoupling sequence. As we'll see, this is possible, at the expense of using more qubits. There will thus be a space-time tradeoff. For an entry into the original literature on this topic see Ref. [12].

### A. Dephasing on two qubits: a hybrid DFS-DD approach

Consider a system consisting of two qubits that are coupled to a bath by the dephasing interaction

$$H_{SB} = \sigma_1^z \otimes B_1^z + \sigma_2^z \otimes B_2^z. \quad (162)$$



This Hamiltonian is not invariant under swapping the two qubits since they couple to different bath operators. To make this more apparent, rewrite the interaction as

$$H_{SB} = \left( \frac{\sigma_1^z - \sigma_2^z}{2} \right) \otimes B_- + \left( \frac{\sigma_1^z + \sigma_2^z}{2} \right) \otimes B_+ \quad (163)$$

where the redefined bath operators are  $B_{\pm} = B_1^z \pm B_2^z$ . We find that  $\left( \frac{\sigma_1^z + \sigma_2^z}{2} \right)$  is a “collective dephasing” operator that applies the same dephasing to both qubits, while the “differential dephasing” operator  $\left( \frac{\sigma_1^z - \sigma_2^z}{2} \right)$  applies opposite dephasing to the two qubits. We now notice that the states  $|01\rangle$  and  $|10\rangle$  (and all superpositions thereof) are annihilated by the collective dephasing operator:

$$(ZI + IZ)(\alpha|01\rangle + \beta|10\rangle) = 0 \quad \forall \quad \alpha, \beta \in \mathbb{C}, \quad (164)$$

where  $ZI = \sigma_1^z \otimes I$  and  $IZ = I \otimes \sigma_2^z$ . These can therefore be chosen as basis elements of a DFS in which a single logical qubit can be encoded as  $|\bar{0}\rangle = |01\rangle$  and  $|\bar{1}\rangle = |10\rangle$ , just as in Eq. (47).

Now that we have chosen a basis that vanishes under the effect of one part of the interaction hamiltonian, this effectively reduces the interaction to

$$H_{SB}|_{\text{DFS}} = \frac{ZI - IZ}{2} \otimes B_- = \bar{\sigma}^z \otimes B_- \quad (165)$$

If the initial interaction had been symmetric, choosing the DFS would have reduced it to zero. However, the interaction was not symmetric in this case, and we are left with the above differential dephasing term. We notice further that the residual term is the same as a  $\bar{\sigma}^z$ , or logical  $Z$  operating on the DFS basis.<sup>2</sup> We recall that dephasing acting on a single qubit was decoupled by pulses that implemented the  $X$  or  $Y$  operators, and hence expect that the  $\bar{\sigma}^z$  interaction can be decoupled using a  $\bar{X}$  or  $\bar{Y}$  pulse. We’ll use the convention that logical/encoded terms in the Hamiltonian are denoted by  $\bar{\sigma}^\alpha$ , while the corresponding unitaries are denoted by  $\bar{X}$ ,  $\bar{Y}$ , or  $\bar{Z}$ . Thus

$$\bar{\sigma}^x = \frac{XX + YY}{2} \quad \bar{\sigma}^y = \frac{YX - XY}{2}. \quad (166)$$

Restricted to the DFS, the implementation of an  $\bar{X}$  pulse using a  $\bar{\sigma}^x$  is analogous to the implementation of an  $X$  pulse by applying  $\sigma^x$  for an appropriate period of time.

$$e^{-i\frac{\pi}{2}\bar{\sigma}^x} = e^{-i\frac{\pi}{4}(XX+YY)} \quad (167a)$$

$$= e^{-i\frac{\pi}{4}XX} e^{-i\frac{\pi}{4}YY} \quad (\text{using } [XX, YY] = 0) \quad (167b)$$

$$= \frac{1}{\sqrt{2}}[I - iXX] \frac{1}{\sqrt{2}}[I - iYY] \quad (\text{using } (XX)^2 = (YY)^2 = I) \quad (167c)$$

$$= \frac{1}{2}[II - iXX - iYY + ZZ] \quad (167d)$$

$$= \frac{-i}{2}(XX + YY) = -i\bar{X}, \quad (167e)$$

where the term  $II + ZZ$  in Eq. (167d) was ignored since it vanishes on the DFS.

Hence, the dynamical decoupling process is effective in the sense that

$$\bar{X} f_\tau \bar{X} f_\tau|_{\text{DFS}} = \exp[-2i\tau(\bar{I} \otimes \tilde{B})] + \mathcal{O}[(2\tau)^2], \quad (168)$$

where  $\tilde{B}$  is a bath operator whose exact form does not matter, since we have obtained a pure-bath operator up to a time  $\mathcal{O}(T^2)$ , where  $T = 2\tau$ . What have we learned from this example? That we don’t need to remove every term in the system-bath Hamiltonian; instead we can use a DFS encoding along with DD. Next we’ll see how this can save us some pulse resources.

## B. General decoherence on two qubits: a hybrid DFS-DD approach

We now consider the most general system-bath Hamiltonian on two qubits:

$$H_{SB} = \sum_{\alpha_1, \alpha_2} (\sigma_1^{\alpha_1} \otimes \sigma_2^{\alpha_2}) \otimes B^{\alpha_1 \alpha_2}, \quad (169)$$

<sup>2</sup> This is a symmetrized version of the logical  $Z$  operator in Eq. (57).

Effect on DFS states	Operators
unchanged	$II, ZI + IZ, ZZ, XX - YY, XY + YX$
logical op.	$\bar{\sigma}^z, \bar{\sigma}^y, \bar{\sigma}^x$
leakage	$XI, IX, YI, IY, XZ, ZX, YZ, ZY$

TABLE I. Classification of all two-qubit error operators on the DFS for collective dephasing.

where  $\alpha_i \in \{0, x, y, z\}$ .

Within the framework of the same DFS as earlier ( $\text{DFS} = \text{span}\{|01\rangle, |10\rangle\}$ ), we can classify all possible ( $4^2 = 16$ ) system operators as either

- leaving system states unchanged (i.e., acting as proportional to  $\bar{I}$ )
- mapping system states to other states within the DFS; these correspond to logical operations (these are errors since they occur as a result of interaction with the bath)
- transitions from the DFS to outside and vice versa (“leakage”)

The operators causing these errors and their effects are given in Table I.

For example,  $ZZ$  acts on  $|\bar{0}\rangle = |01\rangle$  and  $|\bar{1}\rangle = |10\rangle$  as  $-\bar{I}$ , while  $XI$  takes both  $|\bar{0}\rangle$  and  $|\bar{1}\rangle$  out of the DFS, to  $|11\rangle$  and  $|00\rangle$ , respectively.

Along the same lines as single qubit dynamical decoupling, we look for an operator that anticommutes with all the leakage operators, and an operator that anticommutes with all the logical operators. Performing a calculation very similar to Eq. (167) we find that

- $\exp(-i\pi\bar{\sigma}^x) = -ZZ$  and anticommutes with the *entire* leakage set
- $\exp(-i\frac{\pi}{2}\bar{\sigma}^z) = -i\bar{Z}$  and anticommutes with the logical error operators  $\bar{\sigma}^x, \bar{\sigma}^y$ .

A combination of these operators, along with  $\bar{X}$  which we used above, is sufficient to reduce the effect of the system-bath interaction Hamiltonian to that of a pure bath operator that acts trivially on the system. First we apply  $\bar{X}$  to decouple the logical error operators  $\bar{\sigma}^z$  and  $\bar{\sigma}^y$ , giving us a net unitary evolution

$$U_1(2\tau) = \bar{X} f_\tau \bar{X} f_\tau + \mathcal{O}[(2\tau)^2]. \quad (170)$$

We already know that  $U_1$  does not contain  $\bar{\sigma}^z$  and  $\bar{\sigma}^y$  to first order in  $\tau$ , leaving us to compensate for the leakage errors. This is accomplished by using a  $ZZ$  pulse:

$$U_2(4\tau) = ZZ \cdot U_1(2\tau) \cdot ZZ \cdot U_1(2\tau) + \mathcal{O}[(4\tau)^2]. \quad (171)$$

All that remains now is to remove the logical error operator  $\bar{\sigma}^x$ , since it commutes with both the  $\bar{X}$  and  $ZZ$  pulses we have used so far. This can be performed using  $\bar{Z}$ , which anticommutes with  $\bar{\sigma}^x$ . Hence, the overall time evolution that compensates for all possible (logical and leakage) errors is of period  $8\tau$  and is of the form

$$U_3(8\tau) = \bar{Z} U_2(4\tau) \bar{Z} U_2(4\tau) = e^{-8i\tau(\bar{I} \otimes B)} + \mathcal{O}[(8\tau)^2]. \quad (172)$$

A qubit encoded into the  $|\bar{0}\rangle = |01\rangle$  and  $|\bar{1}\rangle = |10\rangle$  DFS is acted on (at time  $T = 8\tau$ ) only by the innocuous operators in the first row of Table I. As a result it is completely free of decoherence, up to errors appearing to  $\mathcal{O}(T^2)$ , while we used a pulse sequence that has length  $8\tau$ , shorter by a factor of 2 compared to the sequence we would have had to use without the DFS encoding (the full two-qubit Pauli group). This, then, illustrates the space-time tradeoff between using full DD without DFS encoding, vs using a hybrid approach, where we use up twice the number of qubits, but gain a factor of two in time.

### IX. CONCATENATED DYNAMICAL DECOUPLING: REMOVING ERRORS OF HIGHER ORDER IN $T$

The dynamical decoupling techniques considered so far have all involved elimination of decoherence up to first order in time. We now consider the question of whether it is possible to further refine these techniques and remove the effect of noise up to higher orders in time. We saw in the earlier sections that applying pulses corresponding to the chosen decoupling group effectively causes a net unitary evolution

$$U_{DD}(T) = \prod_{i=0}^k (g_i^\dagger U_f g_i) = g_0^\dagger U_f g_0 g_1^\dagger U_f g_1 \dots g_k^\dagger U_f g_k \quad (173)$$

where  $U_f$  is the free unitary evolution operator  $U_f = e^{-iH\tau_0}$ ,  $g_i^\dagger U_f g_i = e^{-ig_i^\dagger H g_i \tau_0}$ , and  $T = k\tau_0$ . Without loss of generality, we rewrite  $H$  as

$$H = H_C + H_{NC} \quad (174)$$

where  $H_C$  commutes with the group  $\mathcal{G}$  and  $H_{NC}$  does not. This allows us to simplify the evaluation of  $U_{DD}$ . We proceed by using the Baker-Campbell-Hausdorff formula, which yields

$$U_{DD}^{(1)} = \prod_i g_i^\dagger U_f g_i = \exp \left\{ -i\tau_0 \sum_i g_i^\dagger H g_i + \frac{\tau_0^2}{2} \sum_{i < j} [g_i^\dagger H g_i, g_j^\dagger H g_j] + \mathcal{O}(\tau_0^3) \right\} \quad (175)$$

The first term in the BCH series ( $\mathcal{O}(\tau_0)$  term) can be evaluated as

$$\sum_i g_i^\dagger H g_i = \sum_i g_i^\dagger H_C g_i + \sum_i g_i^\dagger H_{NC} g_i = (k+1)H_C + \sum_i g_i^\dagger H_{NC} g_i \quad (176)$$

We observe that the term in the above equation lies in the centralizer of  $\mathcal{G}$ :

$$g_s^\dagger \left( \sum_i g_i^\dagger H_{NC} g_i \right) g_s = \sum_i g_s^\dagger g_i^\dagger H_{NC} g_s g_i = \sum_j g_j^\dagger H_{NC} g_j \quad (177)$$

We therefore set  $H_C^{(1)} \equiv (k+1)H_C + \sum_j g_j^\dagger H_{NC} g_j$ , which has the property  $[H_C^{(1)}, \mathcal{G}] = 0$ , i.e., the first order correction in the BCH series commutes with all the decoupling group elements. With this redefinition, we can rewrite

$$U_{DD}^{(1)}(T) = \exp \left\{ -iH_C^{(1)}\tau_0 - iH_{NC}^{(1)}\tau_0^2 + \dots \right\} \quad (178)$$

where  $H_{NC}^{(1)}$  comprises the higher order terms (at least  $\mathcal{O}(\tau^2)$ ) of  $H^{(1)}$  which, in general, do not commute with  $\mathcal{G}$ . We now begin the process of concatenation, which essentially amounts to removing the lowest order term in  $\tau$  of  $H_{NC}^{(1)}$  using a second iteration of the group action. This new sequence,  $U_{DD}^{(2)}$  will have an effective Hamiltonian,  $H^{(2)}$  such that  $H_{NC}^{(2)}$  is proportional to  $\mathcal{O}(\tau_0^3)$  or higher. This can be repeated, until the  $m$ th iteration where  $H_{NC}^{(m)}$  is proportional to  $\mathcal{O}(\tau_0^{m+1})$  or higher.

To see this consider  $U_{DD}^{(2)}$ , which is constructed as follows:

$$U_{DD}^{(2)}(kT) = \prod_{i=0}^k g_i^\dagger U_1(T) g_i \quad (179a)$$

$$= \prod_{i=0}^k g_i^\dagger \exp \left\{ -iH_C^{(1)}\tau_0 - iH_{NC}^{(1)}\tau_0^2 + \dots \right\} g_i \quad (179b)$$

$$= \exp \left\{ -i \sum_i g_i^\dagger \left( H_C^{(1)}\tau_0 + H_{NC}^{(1)}\tau_0^2 \right) g_i + \sum_{i < j} \left[ g_i^\dagger (H_C^{(1)}\tau_0 + H_{NC}^{(1)}\tau_0^2) g_i, g_j^\dagger (H_C^{(1)}\tau_0 + H_{NC}^{(1)}\tau_0^2) g_j \right] + \dots \right\} \quad (179c)$$

Looking for terms in the expansion above that do not commute with  $\mathcal{G}$  and are responsible for noise, we find that the lowest order term of this sort is  $\mathcal{O}(\tau_0^3)$ . The part that does commute is labelled  $H_C^{(2)}$  and we get

$$U_{DD}^{(2)}(kT) = e^{-i\{H_C^{(2)}\tau_0 + H_{NC}^{(2)}\tau_0^3 + \mathcal{O}(\tau_0^4)\dots\}} \quad (180)$$

In a similar way, we can obtain  $U_3$  from  $U_2$  as,

$$U_{DD}^{(3)}(k^2T) = e^{-i\{H_C^{(3)}\tau_0 + H_{NC}^{(3)}\tau_0^4 + \mathcal{O}(\tau_0^5)\dots\}}, \quad (181)$$

and in general, the CDD process can be defined and described as

$$U_{DD}^{(m)}(T^{(m)}) := \prod_{i=0}^k g_i^\dagger U_{DD}^{(m-1)}(T^{(m-1)}) g_i \quad (182a)$$

$$= e^{-i\{H_C^{(m)}\tau_0 + H_{NC}^{(m)}\tau_0^{m+1} + \mathcal{O}(\tau_0^{m+2})\dots\}}, \quad (182b)$$

where the total duration of the sequence is given by

$$T^{(m)} = k^m \tau_0. \quad (183)$$

From the above formulation, it seems that we can eliminate the effects of noise on the time evolution of the system to any desired level of accuracy, merely by performing repeated iterations of the dynamical decoupling procedure. Note, however, that in principle the norm of  $H_{NC}^{(m)}$  may grow with  $m$  and thus it is not obvious at this point that increasing orders of CDD concatenation implies a better performance. Let us show that in fact there is an optimal level of concatenation.<sup>3</sup>

Given the unitary evolution generated by  $U_{DD}^{(n)}(T) = e^{-i(H_C^{(n)}\tau_0 + H_{NC}^{(n)}\mathcal{O}(\tau_0^{n+1}))}$  one can define an “error phase”,

$$\phi_{\text{CDD}}(n) = T \|H_{NC}^{(n)}\|, \quad (184)$$

with  $T$  the total time of the sequence. This error phase can be bounded as [13, Eq. (46)]

$$\phi_{\text{CDD}}(n) = T \|H_{NC}^{(n)}\| \leq T(2^{n^2}(\beta\tau_0)^n J), \quad (185)$$

where  $n$  is the degree of concatenation,

$$J = \|H_{SB}\|, \quad \beta = \|H_B\|, \quad (186)$$

and it was assumed that  $\beta \ll 1/T$ . It follows then that for an  $n$ -level CDD sequence we have the following two possibilities:

- Assume a fixed  $T = k^n \tau_0$ , i.e.,  $\tau_0$  can be made arbitrarily small. Then  $\phi_{\text{CDD}}(n) \leq T(2^{n^2}(\beta T/k^n)^n J)$ , and thus the noise strength decreases monotonically as  $\tau_0$  decreases as long as  $(\beta\tau_0(2/k)^n) < 1 \forall n$ . This is in particular true if  $\beta T(2/k) < 1$ . In practice, what this implies is that for a fixed  $T$ , provided  $\beta T$  is small enough, the more concatenations of DD the better.
- On the other hand, if  $\tau_0$  has a minimum physically achievable value, is there an optimal level of concatenation, i.e., a degree of CDD concatenation such that the error is minimized? What is the behavior of  $\phi_{\text{CDD}}(n) \leq k^n \tau_0(2^{n^2}(\beta\tau_0)^n J)$  as  $n$  grows? We see that

$$\begin{aligned} \log(\phi_{\text{CDD}}(n)) &\leq \log\left(k^n \tau_0(2^{n^2}(\beta\tau_0)^n J)\right) \\ &= n \log(k\beta\tau_0) + \log(2)^{n^2} + \log(J\tau_0). \end{aligned} \quad (187)$$

Whenever  $k\beta\tau_0 < 1$ , this expression has a minimum at the optimal concatenation level  $n_{\text{opt}} = -\frac{\log(k\beta\tau_0)}{2\log(2)}$ . This means that for  $n$  larger than  $n_{\text{opt}}$  the CDD process loses its effectiveness, in contrast with the previous case.

## X. DYNAMICAL DECOUPLING AND REPRESENTATION THEORY

In this section, our goal is to illustrate the connections between first-order DD and the result from representation theory, Theorem 2, which is a theorem of fundamental importance in the theory of quantum error correction. See Refs. [11, 14, 15] for entries into the original literature on this topic.

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<sup>3</sup> See Ref. [10] for a rigorous analysis; here we adapt the more intuitive presentation in Ref. [13].

### A. Information storage and computation under DD

We define the *group algebra*  $\mathbb{CG}$  of the group  $\mathcal{G}$  over the complex field as:  $\mathbb{CG} \equiv \{ \text{all linear combinations of the elements in } \mathcal{G}, \text{ over } \mathbb{C} \}$ . For example, if  $\mathcal{G} = \{I, \sigma^x, \sigma^y, \sigma^z\}$ , the Pauli group, then

$$\mathbb{CG} = \left\{ \sum_{\alpha=0,x,y,z} a_\alpha \sigma^\alpha \right\}, \quad a_\alpha \in \mathbb{C}. \quad (188)$$

Now consider the group algebra of our decoupling group,  $\mathbb{CG}$ . Clearly it is a matrix algebra of dimension  $d \times d$  where  $d = \dim(\mathcal{H}_S)$ . We can always choose the group in such a way that  $\mathbb{CG}$  is  $\dagger$ -closed. Since every group includes the identity, we can thus invoke Theorem 2, using which we have:

$$\mathbb{CG} \cong \bigoplus_J I_{n_J} \otimes \mathcal{M}_{d_J}, \quad (189)$$

where  $J$  is the irrep (irreducible representation) label,  $n_J$  is the multiplicity of irrep  $J$  and  $d_J$  is the dimension of the irrep labelled by  $J$ . The system Hilbert space  $\mathcal{H}_S$  is correspondingly partitioned into a direct sum of product spaces, which we can write as

$$\mathcal{H}_S \cong \bigoplus_J \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}. \quad (190)$$

Hence, every DD pulse that we apply to the system acts like identity on  $\mathbb{C}^{n_J}$  and as some non-trivial operation on  $\mathbb{C}^{d_J}$ . It's clear that if we store our quantum information in  $\mathbb{C}^{n_J}$ , then the pulses do not affect it (of course the system-bath interaction can still affect information stored in  $\mathbb{C}^{n_J}$ ).

Now consider the *commutant* of the group algebra  $\mathbb{CG}$  [see Eqs. (98) and (99)],

$$\mathbb{CG}' \equiv \{A | [A, \mathbb{CG}] = 0\} \quad (191a)$$

$$= \bigoplus_J \mathcal{M}_{n_J} \otimes I_{d_J}. \quad (191b)$$

We can immediately see from this definition and from Eq. (159) that, by linearity, the effective system-bath Hamiltonian,  $H'_{SB}$ , lies in the commutant of the group algebra, i.e.,  $H'_{SB} \in \mathbb{CG}'$ . It follows that  $H'_{SB}$  can be represented as:

$$H'_{SB} = \bigoplus_J (H'_{SB})_{n_J} \otimes I_{d_J}, \quad (192)$$

i.e. the effective system-bath Hamiltonian has this block-diagonal representation, with blocks labeled by the irrep index  $J$ , each of dimension  $n_J d_J$ , and where each non-trivial factor  $(H'_{SB})_{n_J}$  is an  $n_J \times n_J$  matrix. It is clear that in this case, we can encode our quantum information in  $\mathbb{C}^{d_J}$ , since  $H'_{SB}$  will act as identity on it. Note that it doesn't necessarily matter that the pulses themselves have a non-trivial effect on  $\mathbb{C}_{d_J}$ : because we know everything about the pulses, they can be compensated for by applying appropriate transformations.

However, it is often desirable not to have to compensate for the action of the DD pulses. In that case we might want to be able to store information in  $\mathbb{C}^{n_J}$  rather than  $\mathbb{C}^{d_J}$ . If so, clearly we need to somehow make the effect of  $H'_{SB}$  trivial on  $\mathbb{C}^{n_J}$  as well. Since  $H'_{SB}$  is determined by our group of pulses,  $\mathcal{G}$ , it boils down to choosing the appropriate set of pulses, i.e., picking the group  $\mathcal{G}$  such that

$$H'_{SB} \in \mathbb{CG}' \cap \mathbb{CG} \quad (193a)$$

$$= \bigoplus_J \lambda_J I_{n_J} \otimes I_{d_J}, \quad \lambda_J \in \mathbb{C} \quad (193b)$$

where Eq. (193b) can be easily deduced by examining Eqs. (189) and (191b). This means that if we pick an appropriate  $\mathcal{G}$ ,  $H'_{SB}$  will have the form:

$$H'_{SB} = \begin{pmatrix} \lambda_1 I_{n_1 d_1} & & & \\ & \lambda_2 I_{n_2 d_2} & & \\ & & \lambda_3 I_{n_3 d_3} & \\ & & & \ddots \end{pmatrix} \quad (194)$$

Suppose in the above:

- We have  $n_J = 1$ , then there would only be a single  $J$  in the direct sum, which would imply that  $d_J = d$ . This would mean that  $H'_{SB} \propto I_d$ , which was the result we obtained by applying Schur's Lemma (1). Thus, that result was a special case resulting from this more general structure.
- If  $\lambda_J = 0$  for all  $J$ , then  $H'_{SB} = 0$  and the system bath interaction is annihilated (to first order in  $T$ ). This describes the kind of situation we obtained with the Klein group symmetrization [see Eq. (148)].

So, now we have established that it is possible to protect information stored both in  $\mathbb{C}^{n_J}$  and  $\mathbb{C}^{d_J}$ . Information stored in the latter is protected since the effective system-bath interaction acts like identity on that space; though we'd have to compensate for the influence of the pulses we would apply. Storing in the former requires us to cleverly choose the pulse group  $\mathcal{G}$  such that the effective system-bath interaction becomes a block-diagonal matrix, with each block proportional to identity. Our choice of storage location might depend on which of  $\max_J n_J$  and  $\max_J d_J$  is greater, since that would provide us with a larger-dimensional space and hence more qubits.

We now discuss some examples where we will use these techniques:

## B. Examples

In all the following examples, our system is a set of  $N$ -qubits, i.e.,  $\mathcal{H}_S = (\mathbb{C}^2)^{\otimes N} = \mathbb{C}^{2^N}$ .

### 1. Example 1: $\mathcal{G} = (SU(2))^{\otimes N}$

In this case the pulses are products of  $N$  arbitrary single-qubit unitaries, and the only operator which commutes with  $\mathcal{G}$  is the identity operator, i.e.,  $H'_{SB} \in \mathbb{C}\mathcal{G}' = \mathbb{C}$ . This is the Schur's Lemma situation again. Thus, in this case we have the choice of storing information in either the left-hand ( $\mathbb{C}^{n_J}$ ) or right-hand factor ( $\mathbb{C}^{d_J}$ ), or both if we do not mind compensating for the action of the pulses on the right-hand factor. This case is interesting since it can be well approximated by picking the  $SU(2)$  rotation on each qubit at random [16].

### 2. Example 2: $\mathcal{G} = \text{Collective } SU(2)$

In this case we have added to Example 1 the constraint that the same unitary matrix acts on every qubit. Then  $H'_{SB} \in \mathbb{C}\mathcal{G}' = \mathbb{C}S_N$ , where  $S_N$  is the permutation group on  $N$ -elements. Thus the effective system-bath Hamiltonian is *not* proportional to identity, and instead we have here the case where  $H'_{SB}$  acts trivially only on the right-hand factors. Its action on the left-hand factors is to apply permutations.

The collective- $SU(2)$  group is generated by the sum of the Pauli matrices, i.e.,  $\{\sum_i \sigma_i^\alpha\}_{\alpha=x,y,z}$ . We already encountered these sums in the study of collective decoherence [see Eqs. (62) and (63)]. In this case, our pulse group has the same generating Hamiltonian as the system part of the system-bath interaction in collective decoherence:  $H_{SB}^{\text{coll.dec.}} = \sum_{\alpha=x,y,z} (\sum_i \sigma_i^\alpha) \otimes B^\alpha$ . Since the DD group acts as identity on the left-hand factors, and the DD group behaves like the collective decoherence operators, while the effective system-bath interaction  $H'_{SB}$  acts like the exchange operators (87) we encountered in our study of computation over the DFS for collective decoherence, we see that the current situation is the reverse (or dual) of the situation back in the DFS case. In other words, we can invoke the machinery we developed then [see Eqs. (100)-(102)] but we should flip the role of  $d_J$  and  $n_J$ :  $\mathcal{H}_S = \bigoplus_{J=0(1/2)}^{N/2} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$ , where now

$$n_J = 2J + 1, \quad (195a)$$

$$d_J = \frac{(2J + 1)N!}{(N/2 + 1 + J)!(N/2 - J)!}. \quad (195b)$$

We can always pick an irrep  $J$  so that  $d_J > n_J$ , and so we have here the same code rates as in the case of a DFS.

### 3. Example 3: $\mathcal{G} = S_n$

In this case  $\mathcal{G}$  is the permutation group. As we saw in the last example, the permutation group is dual to collective  $SU(2)$ . So in this case,  $H'_{SB} \in \mathbb{C}\mathcal{G}' = \text{"collective decoherence"}$ . In fact, since the permutation group can be obtained by swaps (or transpositions), we could as well take  $\mathcal{G} = \{\text{SWAP}_{i,j}\}$ . And we know that,

$$\text{SWAP}_{i,j} = e^{-i\pi \vec{\sigma}_i \cdot \vec{\sigma}_j / 4}, \quad (196)$$

which is generated by the Heisenberg interaction, so the decoupling group is implementable in physical systems (such as quantum dots) where the exchange interaction is controllable (see Ref. [17] for a discussion of how to use efficiently implement  $\mathcal{G}$  in this case). The Hilbert space again splits as  $\mathcal{H}_S = \bigoplus_{J=0(1/2)}^{N/2} \mathbb{C}^{n_J} \otimes \mathbb{C}^{d_J}$ , where now we have the irrep dimension and multiplicity formulas we encountered during the DFS study:

$$n_J = \frac{(2J+1)N!}{(N/2+1+J)!(N/2-J)!} \quad (197a)$$

$$d_J = 2J+1. \quad (197b)$$

We have the option of encoding into the right-hand factor, where  $H'_{SB}$  acts as identity, but then the space dimension is only  $2J+1$ . Alternatively, we can encode into the left-hand factors, where the effective system-bath interaction has non-trivial action, but it acts as collective decoherence, so that our DFS encoding will completely hide the quantum information from the action of  $H'_{SB}$ . This has the significant advantage (over right-hand factor encoding) of providing us with a code space of dimension  $\frac{(2J+1)N!}{(N/2+1+J)!(N/2-J)!}$ . This example leads us to the interesting conclusion that in this case in fact Eq. (193b) applies, i.e. the effective system-bath interaction acts trivially everywhere.

#### 4. Example 4: Linear System-Bath coupling

We consider a system-bath interaction of the form,

$$H_{SB} = \sum_{\alpha=x,y,z} \sum_i \sigma_i^\alpha \otimes B_i^\alpha. \quad (198)$$

Each qubit in this case has its own bath. In this noise model, we don't consider bilinear terms in the system such as  $\sigma_i^\alpha \sigma_j^\beta \otimes B_{ij}^{\alpha\beta}$ , because this is a 3-body interaction which is typically much weaker in nature and also very hard to engineer.

The decoupling group we select for this is,

$$\mathcal{G} = \{I^{\otimes N}, X^{\otimes N}, Y^{\otimes N}, Z^{\otimes N}\}. \quad (199)$$

We choose  $N$  to be even. Therefore,  $\mathcal{G}$  becomes abelian. And from representation theory, we know that all the irreps of an abelian group are 1-dimensional (scalars, so  $d_J = 1 \forall J$ ) and the number of irreps is the order of the group. Here  $|\mathcal{G}| = 4$ . The irreps are:

$J$	$I^{\otimes N}$	$X^{\otimes N}$	$Y^{\otimes N}$	$Z^{\otimes N}$
1	1	1	1	1
2	1	1	-1	-1
3	1	-1	1	-1
4	1	-1	-1	1

The group algebra is,

$$\mathbb{C}\mathcal{G} \cong \bigoplus_J I_{n_J} \otimes \mathcal{M}_{d_J} \quad (200a)$$

$$= \bigoplus_{J=1}^4 c_J I_{2^{N-2}}, \quad (200b)$$

where in Eq. (200b)  $n_J = 2^{N-2}$  because  $\sum_J n_J d_J = 2^N$  and all the  $n_J$ 's are equal (by use of a standard multiplicity formula from group theory: exercise for the reader), we have  $4n_J = 2^N$ , making  $n_J = 2^{N-2}$ .

Thus the group algebra can be represented as a block diagonal matrix, each block being proportional to an identity of dimension  $2^{N-2}$  with proportionality constant  $c_J$ , i.e., we have  $(N-2)$  qubits in each block that will be unaffected by the system-bath interaction.

Let us pick the first (trivial) irrep to encode our qubits into, i.e.,  $\{1, 1, 1, 1\}$ . In this irrep each pulse acts as 1, so we're looking for code-states which are "stabilized" by the group (each group element acts as identity). After a bit of thought it is clear that such states are of the form:

$$|\psi_r\rangle \equiv \frac{1}{\sqrt{2}} (|r\rangle + |\bar{r}\rangle), \quad (201)$$

where  $\bar{r} = \text{NOT}(r)$  and  $r \in \{0, 1\}^N$  is an *even* weight binary string of  $N$  bits, i.e.,  $r$  contains an even number of 1's. Then it is easy to see that the action of any member of the decoupling group  $\mathcal{G}$ , leaves  $|\psi_r\rangle$  unchanged, so indeed  $|\psi_r\rangle$  belongs to the trivial irrep.

Why did we pick this decoupling group? Because it has some very interesting and useful features which we list:

*Feature 1:* We can show that  $H'_{SB} = 0$ .

*Proof.*

$$H'_{SB} = \frac{1}{|\mathcal{G}|} \sum_{j=0}^{|\mathcal{G}|} g_j^\dagger H'_{SB} g_j \quad (202a)$$

$$= \frac{1}{4} [IH_{SB}I + X^{\otimes N} H_{SB} X^{\otimes N} + Y^{\otimes N} H_{SB} Y^{\otimes N} + Z^{\otimes N} H_{SB} Z^{\otimes N}] \quad (202b)$$

$$= \frac{1}{4} \left[ H_{SB} + \sum_i \left( \sigma_i^X \otimes B_i^X - \sigma_i^Y \otimes B_i^Y - \sigma_i^Z \otimes B_i^Z \right. \right. \\ \left. \left. - \sigma_i^X \otimes B_i^X + \sigma_i^Y \otimes B_i^Y - \sigma_i^Z \otimes B_i^Z \right. \right. \\ \left. \left. - \sigma_i^X \otimes B_i^X - \sigma_i^Y \otimes B_i^Y + \sigma_i^Z \otimes B_i^Z \right) \right] \quad (202c)$$

$$= 0, \quad (202d)$$

where to arrive at Eq. (202c) we used the properties of the Pauli group to do the multiplication. For example,

$$X^{\otimes N} \sigma_i^Y X^{\otimes N} = (\sigma_1^X \otimes \dots \otimes \sigma_i^X \otimes \dots \otimes \sigma_N^X) (\sigma_i^Y) (\sigma_1^X \otimes \dots \otimes \sigma_i^X \otimes \dots \otimes \sigma_N^X) \quad (203a)$$

$$= -\sigma_i^Y, \quad (203b)$$

and so on.  $\square$

Thus, this decoupling group eliminates the system-bath interaction completely, to first order. Recall that this means that we can use the left-hand factor to encode and store information. This allows us to perform computation using the commutant, which has non-trivial action on the left-hand factor.

*Feature 2:* We can do computation on the decoherence-protected qubits.

For  $N = 2$ , i.e., for 2 physical qubits, we have no logical qubits, as the only one state possible according to Eq. (201) is  $|\psi_r\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ . This agrees with the fact that, since each irrep provides us with  $N - 2$  logical qubits, we have zero logical qubits for 2 physical qubits.

Let us list the possible states in the case of  $N = 4$ , using Eq. (201)

$$|\psi_{0000}\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle) \equiv |\bar{0}\bar{0}\rangle, \quad (204a)$$

$$|\psi_{0011}\rangle = \frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle) \equiv |\bar{1}\bar{0}\rangle, \quad (204b)$$

$$|\psi_{0101}\rangle = \frac{1}{\sqrt{2}}(|0101\rangle + |1010\rangle) \equiv |\bar{0}\bar{1}\rangle, \quad (204c)$$

$$|\psi_{0110}\rangle = \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle) \equiv |\bar{1}\bar{1}\rangle, \quad (204d)$$

and since we have 4 orthonormal states, we can use them as 2 qubits. Again, this agrees with the fact that  $N - 2 = 2$  in this case. In the above equations, the logical states have a bar on top of their labels.

How do we perform computations on these states? For that we use the commutant of the group,

$$\mathbb{C}\mathcal{G}' \cong \bigoplus \mathcal{M}_{n_J} \otimes I_{d_J} \quad (205a)$$

$$= \bigoplus_{J=1}^4 c_J \mathcal{M}_{2^{N-2}} \quad (205b)$$

where  $c_J$  are scalars. We can check that the commutant can be generated by  $\{X_1 X_{j+1}\}_{j=1}^{N-2} \cup \{Z_{j+1} Z_N\}_{j=1}^{N-2}$ . For example, we don't require  $Y_i Y_j$  to be in the generating set, since  $Y_i Y_j = X_i X_j Z_i Z_j$ . Therefore, for  $N = 4$ , the generating set for the commutant becomes  $\{X_1 X_2, X_1 X_3, Z_2 Z_4, Z_3 Z_4\}$ .



Let's check the action of  $X_1 X_2$  on our first logical state  $|\psi_{0000}\rangle \equiv |\bar{0}\bar{0}\rangle$ ,

$$X_1 X_2 |\bar{0}\bar{0}\rangle = (X_1 \otimes X_2 \otimes I \otimes I) \left( \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) \right) \quad (206a)$$

$$= \frac{1}{\sqrt{2}} (|0011\rangle + |1100\rangle) \equiv |\bar{1}\bar{0}\rangle. \quad (206b)$$

We can similarly check that  $X_1 X_2 |\bar{0}\bar{1}\rangle = |\bar{1}\bar{1}\rangle$ . Therefore  $X_1 X_2$  is logical Pauli- $X$  on the first logical qubit. We write this as  $X_1 X_2 = \bar{X}_1$ , where the bar denotes a logical operator. Similarly we can verify that:

$$X_1 X_{j+1} = \bar{X}_j, \quad (207)$$

$$Z_{j+1} Z_N = \bar{Z}_j. \quad (208)$$

And having obtained logical- $X$  and logical- $Z$  (note that they anti-commute, as they should), we can implement any one qubit gate we like using only two-body interactions. And moreover, these logical gates lie in the commutant of the decoupling group. Therefore, we can compute while at the same time applying DD.

But, in order to perform universal quantum computation, we also need to be able to perform entangling operations on two qubits, for example the CNOT gate. Let's examine  $\bar{X}_i \bar{X}_j$  which is a logical entangling operation :

$$\bar{X}_i \bar{X}_j = X_1 X_{i+1} X_1 X_{j+1} \quad (209a)$$

$$= X_{i+1} X_{j+1}, \quad (209b)$$

which is a physical 2-body interaction. So, we have managed to implement a logical entangling operation using only a two-body interaction.

It is well known that if we can implement any Hamiltonian of the form,

$$H_S = \sum_i \omega_i^X(t) \sigma_i^X + \sum_i \omega_i^Z(t) \sigma_i^Z + \sum_{\alpha \in \{X, Z\}} \sum_{i,j} J_{ij}(t) \sigma_i^\alpha \sigma_j^\alpha, \quad (210)$$

then we can perform universal quantum computation [1]. Therefore, in our case, replacing the Pauli operators in Eq. (210) with their logical counterparts and expanding them in terms of their decompositions (207) in the physical qubit space, we obtain:

$$\bar{H}_S = \sum_{i=1}^{N-2} \bar{\omega}_i^X(t) \bar{\sigma}_i^X + \sum_{i=1}^{N-2} \bar{\omega}_i^Z(t) \bar{\sigma}_i^Z + \sum_{\alpha \in \{X, Z\}} \sum_{i,j} \bar{J}_{ij}(t) \bar{\sigma}_i^\alpha \bar{\sigma}_j^\alpha \quad (211a)$$

$$= \sum_{i=1}^{N-2} \omega_{i+1}^X(t) \sigma_{i+1}^x \sigma_{i+1}^x + \sum_{i=1}^{N-2} \omega_{i+1}^Z(t) \sigma_{i+1}^z \sigma_N^z + \sum_{\alpha \in \{X, Z\}} \sum_{i,j} J_{i+1,j+1}(t) \sigma_{i+1}^\alpha \sigma_{j+1}^\alpha \quad (211b)$$

Remarkably, we have obtained *decoupling and universal quantum computation* on the logical qubits using only two-body interactions on the physical qubits. A similar approach has been studied numerically in the context of the 4-qubit DFS code, with gates protected by CDD, showing evidence for a highly robust set of universal gates [18]. Related ideas apply in the context of adiabatic quantum computation [19].

## XI. CONCLUSIONS

This review has covered a selection of topics in the theory of decoherence-free subspaces, noiseless subsystems, and dynamical decoupling. We have seen how these tools allow one to hide information from the environment, and when this hiding is imperfect, how dynamical decoupling allows us to suppress the remaining residual decoherence. Moreover, we have shown explicitly how universal quantum computation is compatible with decoherence avoidance and suppression.

Many important topics were left out in this brief review. For example, apart from CDD we did not address high-order decoupling methods, in particular schemes based on optimized pulse intervals [20–23]. Nor did we address the filter function approach to DD [24], optimized continuous modulation [7, 25], or randomized decoupling, which is well suited to strongly time-dependent baths [26]. It is important to stress that beyond decoherence avoidance and suppression, the theory of noiseless subsystems gave rise also to important advances in the theory of quantum error correcting codes, such as operator quantum error correction [27]. However, perhaps our greatest omission has been the abundance of experimental results which have both confirmed and driven the theoretical developments described here. Nevertheless, hopefully we have given the reader the tools and inspiration to delve deeper into the large and fascinating literature on decoherence avoidance and suppression.

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- [1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2000).
  - [2] D. A. Lidar and K. B. Whaley, *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini, Lecture Notes in Physics, Vol. 622 (Springer Berlin / Heidelberg, 2003) pp. 83–120.
  - [3] W. Yang, Z.-Y. Wang, and R.-B. Liu, *Front. Phys.*, **6**, 2 (2011).
  - [4] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover Publications, 1989).
  - [5] D. Bacon, J. Kempe, D. A. Lidar, and K. B. Whaley, *Phys. Rev. Lett.*, **85**, 1758 (2000).
  - [6] E. Knill, R. Laflamme, and L. Viola, *Phys. Rev. Lett.*, **84**, 2525 (2000).
  - [7] G. Gordon, G. Kurizki, and D. A. Lidar, *Phys. Rev. Lett.*, **101**, 010403 (2008).
  - [8] D. S. Bernstein, *Matrix Mathematics* (Princeton University Press, Princeton, 2005).
  - [9] R. Bhatia, *Matrix Analysis*, Graduate Texts in Mathematics No. 169 (Springer-Verlag, New York, 1997).
  - [10] H. K. Ng, D. A. Lidar, and J. Preskill, *Phys. Rev. A*, **84**, 012305 (2011).
  - [11] P. Zanardi, *Phys. Lett. A*, **258**, 77 (1999).
  - [12] M. S. Byrd and D. A. Lidar, *Phys. Rev. Lett.*, **89**, 047901 (2002).
  - [13] K. Khodjasteh and D. A. Lidar, *Phys. Rev. A*, **75**, 062310 (2007).
  - [14] L. Viola, S. Lloyd, and E. Knill, *Phys. Rev. Lett.*, **83**, 4888 (1999).
  - [15] L. Viola, E. Knill, and S. Lloyd, *Phys. Rev. Lett.*, **85**, 3520 (2000).
  - [16] L. Viola and E. Knill, *Phys. Rev. Lett.*, **94**, 060502 (2005).
  - [17] L.-A. Wu and D. A. Lidar, *Phys. Rev. Lett.*, **88**, 207902 (2002).
  - [18] J. R. West, D. A. Lidar, B. H. Fong, and M. F. Gyure, *Phys. Rev. Lett.*, **105**, 230503 (2010).
  - [19] D. A. Lidar, *Phys. Rev. Lett.*, **100**, 160506 (2008).
  - [20] G. S. Uhrig, *Phys. Rev. Lett.*, **98**, 100504 (2007).
  - [21] J. R. West, B. H. Fong, and D. A. Lidar, *Phys. Rev. Lett.*, **104**, 130501 (2010).
  - [22] Z.-Y. Wang and R.-B. Liu, *Phys. Rev. A*, **83**, 022306 (2011).
  - [23] Y. Xia, G. S. Uhrig, and D. A. Lidar, *Phys. Rev. A*, **84**, 062332 (2011).
  - [24] H. Uys, M. J. Biercuk, and J. J. Bollinger, *Phys. Rev. Lett.*, **103**, 040501 (2009).
  - [25] J. Clausen, G. Bensky, and G. Kurizki, *Phys. Rev. Lett.*, **104**, 040401 (2010).
  - [26] L. F. Santos and L. Viola, *Phys. Rev. Lett.*, **97**, 150501 (2006).
  - [27] D. Kribs, R. Laflamme, and D. Poulin, *Phys. Rev. Lett.*, **94**, 180501 (2005).